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Estimates of the domain of dependence for scalar conservation laws

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Abstract

We consider the Cauchy problem for a multidimensional scalar conservation law and construct an outer estimate for the domain of dependence of its Kružkov solution. The estimate can be represented as the controllability set of a specific differential inclusion. In addition, reachable sets of this inclusion provide outer estimates for the support of the wave profiles. Both results follow from a modified version of the classical Kružkov uniqueness theorem, which we also present in the paper. Finally, the results are applied to a control problem consisting in steering a distributed quantity to a given set.

MSC: 35L65; 93B03

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1. Introduction

The paper aims at constructing an outer estimate for the domain of dependence of the Kružkov solution to the following Cauchy problem

$$\partial_t u + \operatorname{div}\left(f(t, x, u)\right) = 0, \tag{1}$$

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$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^n. \tag{2}$$

Recall the precise definition of the domain of dependence.

Definition 1.1 *(D. Serre [23]).* Let u be a Kružkov solution to (1), (2). The *domain of dependence* of u at a point (t,x) is the smallest compact set $\mathcal{D}_u(t,x) \subset \mathbb{R}^n$ such that, for every bounded function w with compact support disjoint from $\mathcal{D}_u(t,x)$ and every positive ε small enough, the solution of the Cauchy problem with the initial condition $v_0 = u_0 + \varepsilon w$ coincides with u at (t,x).

Two things concerning this definition must be clarified. First of all, since the initial function u_0 usually belongs to $\mathbf{L}^\infty(\mathbb{R}^n)$, by the support of u_0 we mean the support of the singed measure $u_0\mathcal{L}^n$, i.e., the smallest closed set A such that $u_0(x) = 0$ almost everywhere (a.e.) on its complement A^c. Throughout the paper, this set is denoted by $\operatorname{spt} u_0$.

Another issue follows from the fact that a Kružkov solution is defined as a map of class $\mathbf{C}^0\left([0,\infty); \mathbf{L}^1_{loc}(\mathbb{R}^n)\right)$. By saying that two solutions u and v coincide at (t,x), we mean that

$$\lim_{r \to 0} \frac{1}{\mathcal{L}^n\left(B(x,r)\right)} \int_{B(x,r)} |u(t,y) - v(t,y)| \, \mathrm{d}y = 0, \tag{3}$$

where $B(x,r)$ denotes the closed ball of radius r centered at x.

For a very special case of (1), when the equation is one-dimensional and the flow is convex in u, the domain of dependence can be found explicitly by the method of generalized characteristics [9, Chapter 10]. In the general case, a rough outer estimate is provided by the Kružkov uniqueness theorem [15]:

$$\mathcal{D}_u(t,x) \subseteq B(x,ct),$$

where $c = \sup\{|\partial_u f(s,x,u)| : \ s \in [0,t], \ x \in \mathbb{R}^n, \ u \in \mathbb{R}\}$.

Note that the ball $B(x,ct)$ is exactly *the controllability set* of the differential inclusion

$$\dot{y}(s) \in B(0,c), \tag{4}$$

i.e., the set of all points $a \in \mathbb{R}^n$ that can be connected with x by a trajectory $y \colon [0,t] \to \mathbb{R}^n$ of (4).

Encouraged by this observation, we are going to find a differential inclusion

$$\dot{y}(s) \in F\left(s, y(s)\right), \tag{5}$$

whose right-hand side is smaller than $B(0,c)$ and whose controllability set still gives an outer estimate of $\mathcal{D}_u(t,x)$. As we shall see, a possible choice for such set-valued map F is

$$F(s,y) = \operatorname{co} \partial_u f\left(s, y, [a(s), b(s)]\right),$$

where a and b are certain upper and lower bounds of u, while 'co' denotes the convex hull of a set.

To check that the controllability set contains the domain of dependence, we use a modified version of the classical Kružkov theorem. In this version the cone appearing in the original theorem is substituted by the backward integral funnel of (5). A difficult moment appears at this