ARTICLE IN PRESS YJDEQ:9287

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J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Continuity and minimization of spectrum related with the periodic Camassa–Holm equation ☆

Jifeng Chu [a,*], Gang Meng [b], Meirong Zhang [c]

[a] *Department of Mathematics, Shanghai Normal University, Shanghai 200234, China*

[b] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*

[c] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China*

Received 6 February 2018; revised 17 March 2018

Abstract

An important point in looking for period solutions of the Camassa–Holm equation is to understand the associated spectral problem

$$y'' = \frac{1}{4}y + \lambda m(t)y.$$

The first aim of this paper is to study the dependence of eigenvalues for the periodic Camassa–Holm Equation on potentials as an infinitely dimensional parameter. To be precise, we prove that as nonlinear functionals of potentials, eigenvalues for the periodic Camassa–Holm Equation are continuous in potentials with respect to the weak topologies in the L^p Lebesgue spaces. The second aim of this paper is to find the optimal lower bound of the lowest eigenvalue for the periodic Camassa–Holm Equation when the L^1 norm of potentials are given. In order to make our results more applicable, we will find the optimal lower bound for the lowest eigenvalue in the more general setting of measure differential equations.

MSC: primary 34L15, 34L40, 76B15

☆ Jifeng Chu was supported by the National Natural Science Foundation of China (Grant No. 11671118). Gang Meng was supported by the National Natural Science Foundation of China (Grant No. 11671378) and the Fund of UCAS. Meirong Zhang was supported by the National Natural Science Foundation of China (Grant No. 11790273 and No. 11371213).

* Corresponding author.

E-mail addresses: jchu@shnu.edu.cn, jifengchu@126.com (J. Chu), menggang@ucas.ac.cn (G. Meng), mzhang@math.tsinghua.edu.cn (M. Zhang).

https://doi.org/10.1016/j.jde.2018.04.016

0022-0396/

Please cite this article in press as: J. Chu et al., Continuity and minimization of spectrum related with the periodic Camassa–Holm equation, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.04.016

Keywords: Eigenvalue; Continuity; Minimization; Camassa–Holm equation; Measure differential equations

1. Introduction

The Camassa–Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1.1}$$

was derived in [2,3] to describe the motion of solitary waves on shallow water. During the last two decades, many important and interesting results have been obtained for the Camassa–Holm equation. See [1,2,7,8,14,17,21] and the references therein. In looking for spatially periodic solutions of (1.1), a key point is to understand the associated spectral problem

$$y'' = \frac{1}{4}y + \lambda m(t)y, \tag{1.2}$$

with the periodic boundary condition

$$y(0) = y(1), \quad y'(0) = y'(1), \tag{1.3}$$

where $m = u - u_{xx}$ is the potential. The spectrum of (1.2)–(1.3) is the set of those values of $\lambda \in \mathbb{R}$ for which problem (1.2)–(1.3) has a nontrivial solution. While the solution of the Camassa–Holm equation (1.1) evolves in a complicated way, the periodic (and antiperiodic) spectra remain unchanged (see the discussions in [2]). A quite complete picture of the spectrum for $m \in \mathcal{C}^1[0,1]$ was given in [10]. Using a different method from that in [10], it was shown in [6] that the results in [10] can be adapted to the general and natural case $m \in \mathcal{C} := \mathcal{C}[0,1]$, the space of continuous functions on $[0,1]$. More information about the oscillation of eigenfunctions was also obtained in [6]. It was shown in [9] that the potential can also be constructed from the periodic spectrum. We also refer to [11–13] for recent results on the isospectral problems and the corresponding inverse spectral problems of the Camassa–Holm equation.

In this paper, we consider the problem (1.2)–(1.3) for with $m \in \mathcal{C}^-$, where

$$\mathcal{C}^- = \{m \in \mathcal{C} : m(t) \le 0,\ m(t) \not\equiv 0\}.$$

In this case, it follows from [6] that problem (1.2)–(1.3) has a sequence of eigenvalues

$$0 < \lambda_0(m) < \lambda_1(m) \le \lambda_2(m) < \cdots < \lambda_{2k-1}(m) \le \lambda_{2k}(m) < \cdots,$$

satisfying $\lim_{k\to\infty} \lambda_k(m) = +\infty$. Moreover, the lowest eigenvalue $\lambda_0(m)$ is simple. However, for $k \in \mathbb{N}$, $\lambda_{2k-1}(m)$ and $\lambda_{2k}(m)$ may have multiplicity of 2.

The first aim of this paper is to study the dependence of eigenvalues $\{\lambda_k(m)\}$ on potentials m as an infinitely dimensional parameter. To be precise, we will prove that for each $k \in \mathbb{Z}^+ := \{0\} \cup \mathbb{N}$, as a nonlinear functional of potentials $m \in \mathcal{C}^-$, eigenvalue $\lambda_k(m)$ is continuous in m when the weak topology w_1 inherited from the weak topology of the Lebesgue space $\mathcal{L}^1 := \mathcal{L}^1([0,1])$ is considered. See [4,22] for the weak topology of the Lebesgue space $\mathcal{L}^1$. For the precise statement