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Symmetric Liapunov center theorem for minimal orbit

Ernesto Pérez-Chavela [a], Sławomir Rybicki [b,∗], Daniel Strzelecki [b]

[a] Departamento de Matemáticas, Instituto Tecnologico Autónomo de México (ITAM), Ró Hondo 1, Col. Progreso Tizapán México D.F. 01080, Mexico

[b] Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, PL-87-100 Toruń ul. Chopina 12/18, Poland

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Abstract

Using the techniques of equivariant bifurcation theory we prove the existence of non-stationary periodic solutions of Γ-symmetric systems $\ddot{q}(t) = -\nabla U(q(t))$ in any neighborhood of an isolated orbit of minima $\Gamma(q_0)$ of the potential U. We show the strength of our result by proving the existence of new families of periodic orbits in the Lennard-Jones two- and three-body problems and in the Schwarzschild three-body problem.

MSC: primary 37G15; secondary 37G40

Keywords: Periodic solutions; Equivariant Conley index; Liapunov center theorem

1. Introduction

The study of the existence of non-stationary periodic solutions of autonomous ordinary differential equations has a long history. A particular attention was paid to the study of the existence of such solutions in a neighborhood of isolated equilibria, see for instance [12,17,21,22,25,29] and references therein. Of course this list is far from being complete.

∗ Corresponding author.

E-mail addresses: ernesto.perez@itam.mx (E. Pérez-Chavela), rybicki@mat.umk.pl (S. Rybicki), danio@mat.umk.pl (D. Strzelecki).

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One of the most famous theorems concerning the existence of periodic solutions of ordinary differential equations is the celebrated Liapunov center theorem.

Consider a second order system $\ddot{q}(t) = -\nabla U(q(t))$, where $U \in C^2(\mathbb{R}^n, \mathbb{R})$, $\nabla U(0) = 0$ and $\det \nabla^2 U(0) \neq 0$. Let $\sigma(\nabla^2 U(0))$ be the spectrum of the Hessian $\nabla^2 U(0)$. The Liapunov center theorem says that if $\sigma(\nabla^2 U(0)) \cap (0, +\infty) = \{\beta_1^2, \ldots, \beta_m^2\}$ for $\beta_1 > \ldots > \beta_m > 0$ then for β_{j_0} satisfying $\beta_1/\beta_{j_0}, \ldots, \beta_{j_0-1}/\beta_{j_0} \notin \mathbb{N}$, there is a sequence $\{q_k(t)\}$ of periodic solutions of the system

$$\ddot{q}(t) = -\nabla U(q(t)), \tag{1.1}$$

with amplitude tending to zero and the minimal period tending to $2\pi/\beta_{j_0}$. The proof of this theorem can be found in [22], see also [5,6]. Generalizations of the Liapunov center theorem were developed in many directions. In [5,6,10,26,34] one can find some of them.

Let $\Omega \subset \mathbb{R}^n$ be an open and Γ-invariant subset of $\mathbb{R}^n$ considered as a representation of a compact Lie group Γ. Assume that $q_0 \in \Omega$ is a critical point of the Γ-invariant potential $U : \Omega \to \mathbb{R}$ of class C^2. Since for all $\gamma \in \Gamma$ the equality $U(\gamma q_0) = U(q_0)$ holds and $\nabla U(q_0) = 0$, the orbit $\Gamma(q_0) = \{\gamma q_0 : \gamma \in \Gamma\}$ consists of critical points of U i.e. $\Gamma(q_0) \subset (\nabla U)^{-1}(0)$. Note that if $\dim \Gamma \geqq 1$ then it can happen that $\dim \Gamma(q_0) \geqq 1$ i.e. the critical point q_0 is not isolated in $(\nabla U)^{-1}(0)$. That is why for higher-dimensional orbits $\Gamma(q_0)$ we can not apply the classical Liapunov center theorem.

In [26] we have proved the Symmetric Liapunov center theorem for a non-degenerate orbit of critical points $\Gamma(q_0)$ i.e. we have assumed that $\dim \Gamma(q_0) = \dim \ker \nabla^2 U(q_0)$. More precisely, with the additional hypothesis that the isotropy group $\Gamma_{q_0} = \{\gamma \in \Gamma : \gamma q_0 = q_0\}$ is trivial and that there is at least one positive eigenvalue of the Hessian $\nabla^2 U(q_0)$, we have proved the existence of non-stationary periodic solutions of the system (1.1) in any neighborhood of the orbit $\Gamma(q_0)$. Moreover, we are able to control the minimal period of these solutions in terms of the positive eigenvalues of $\nabla^2 U(q_0)$.

For the Lennard-Jones and Schwarzschild problems discussed in the last section there are isolated degenerate circles ($\Gamma = SO(2)$-orbits) of stationary solutions which consist of minima of the corresponding potentials. We underline that we are not able to study the non-stationary periodic solutions of these problems applying the classical Liapunov center theorem because these equilibria are not isolated. We also emphasize that since these orbits are degenerate, we either can not study the non-stationary periodic solutions of these problems applying the Symmetric Liapunov center theorem for non-degenerate orbit proved in [26]. Therefore there is a natural need to prove the Symmetric Liapunov center theorem for isolated orbits of minima.

The inspiration for writing this article, in addition to the discussion above, was a nice paper of Rabinowitz [27], where the author proved that the Brouwer index of an isolated minimum of a potential of the class C^1 is equal to 1. This result was also proved later by Amann [2].

The goal of this paper is to prove the Symmetric Liapunov center theorem for an isolated orbit of minima of the potential U. Our main result is the following.

Theorem 1.1. *[Symmetric Liapunov center theorem for a minimal orbit] Let $U : \Omega \to \mathbb{R}$ be a Γ-invariant potential of the class C^2 and $q_0 \in \Omega$. Assume that*

(1) *the orbit $\Gamma(q_0)$ consists of minima of potential U,*
(2) *the orbit $\Gamma(q_0)$ is isolated in $(\nabla U)^{-1}(0)$,*
(3) *the isotropy group Γ_{q_0} is trivial,*
(4) *$\sigma(\nabla^2 U(q_0)) \cap (0, +\infty) = \{\beta_1^2, \ldots, \beta_m^2\}$, $\beta_1 > \beta_2 > \ldots > \beta_m > 0$ and $m \geq 1$.*

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