ARTICLE IN PRESS YJDEQ:9257

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Improvements on lower bounds for the blow-up time under local nonlinear Neumann conditions

Xin Yang [a,*], Zhengfang Zhou [b]

[a] *Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, United States*
[b] *Department of Mathematics, Michigan State University, East Lansing, MI 48824, United States*

Received 6 July 2017; revised 10 March 2018

Abstract

This paper studies the heat equation $u_t = \Delta u$ in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ with positive initial data and a local nonlinear Neumann boundary condition: the normal derivative $\partial u/\partial n = u^q$ on partial boundary $\Gamma_1 \subseteq \partial\Omega$ for some $q > 1$, while $\partial u/\partial n = 0$ on the other part. We investigate the lower bound of the blow-up time T^* of u in several aspects. First, T^* is proved to be at least of order $(q-1)^{-1}$ as $q \to 1^+$. Since the existing upper bound is of order $(q-1)^{-1}$, this result is sharp. Secondly, if Ω is convex and $|\Gamma_1|$ denotes the surface area of Γ_1, then T^* is shown to be at least of order $|\Gamma_1|^{-\frac{1}{n-1}}$ for $n \geq 3$ and $|\Gamma_1|^{-1} / \ln\left(|\Gamma_1|^{-1}\right)$ for $n = 2$ as $|\Gamma_1| \to 0$, while the previous result is $|\Gamma_1|^{-\alpha}$ for any $\alpha < \frac{1}{n-1}$. Finally, we generalize the results for convex domains to the domains with only local convexity near Γ_1.

MSC: 35B44; 35K60; 35K05; 35C15

Keywords: Blow-up time; Lower bound; Local nonlinear Neumann boundary condition

* Corresponding author.
E-mail addresses: yang2x2@ucmail.uc.edu (X. Yang), zfzhou@math.msu.edu (Z. Zhou).

https://doi.org/10.1016/j.jde.2018.03.013
0022-0396/

Please cite this article in press as: X. Yang, Z. Zhou, Improvements on lower bounds for the blow-up time under local nonlinear Neumann conditions, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.03.013

1. Introduction

1.1. Problem and notations

In this paper, unless otherwise stated, Ω represents a bounded open subset in $\mathbb{R}^n$ ($n \geq 2$) with C^2 boundary $\partial\Omega$. Γ_1 and Γ_2 denote two disjoint relatively open subsets of $\partial\Omega$. $\partial\Gamma_1 = \partial\Gamma_2 \triangleq \widetilde{\Gamma}$ is a common C^1 boundary of Γ_1 and Γ_2. Moreover, $\Gamma_1 \neq \emptyset$ and $\partial\Omega = \Gamma_1 \cup \widetilde{\Gamma} \cup \Gamma_2$. We study the following problem:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = u^q(x,t) & \text{on} \quad \Gamma_1 \times (0,T], \\ \frac{\partial u(x,t)}{\partial n(x)} = 0 & \text{on} \quad \Gamma_2 \times (0,T], \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where

$$q > 1,\ u_0 \in C^1(\overline{\Omega}),\ u_0(x) \geq 0,\ u_0(x) \not\equiv 0. \tag{1.2}$$

The normal derivative in (1.1) is understood in the following way: for any $(x,t) \in \partial\Omega \times (0,T]$,

$$\frac{\partial u(x,t)}{\partial n(x)} \triangleq \lim_{h \to 0^+} (Du)(x_h, t) \cdot \overrightarrow{n}(x), \tag{1.3}$$

where Du denotes the spatial derivative of u, $\overrightarrow{n}(x)$ denotes the exterior unit normal vector at x and $x_h \triangleq x - h\overrightarrow{n}(x)$ for $x \in \partial\Omega$. Since $\partial\Omega$ is C^2, x_h belongs to Ω when h is positive and sufficiently small.

Throughout this paper, we write

$$M_0 = \max_{x \in \overline{\Omega}} u_0(x) \tag{1.4}$$

and denote $M(t)$ to be the supremum of the solution u to (1.1) on $\overline{\Omega} \times [0,t]$:

$$M(t) = \sup_{(x,\tau) \in \overline{\Omega} \times [0,t]} u(x,\tau). \tag{1.5}$$

$|\Gamma_1|$ represents the surface area of Γ_1, that is

$$|\Gamma_1| = \int_{\Gamma_1} dS(x),$$

where $dS(x)$ means the surface integral with respect to the variable x. Φ refers to the fundamental solution to the heat equation:

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\Big(-\frac{|x|^2}{4t}\Big), \quad \forall (x,t) \in \mathbb{R}^n \times (0,\infty). \tag{1.6}$$

Please cite this article in press as: X. Yang, Z. Zhou, Improvements on lower bounds for the blow-up time under local nonlinear Neumann conditions, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.03.013