Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 265 (2018) 906–920

Journal of Differential Equations

www.elsevier.com/locate/jde

On the Gevrey regularity for sums of squares of vector fields, study of some models

Gregorio Chinni *

University of Vienna, Fakultät für Mathematik, Austria

Received 23 June 2017; revised 21 January 2018
Available online 17 April 2018

Abstract

The Gevrey hypoellipticity of a class of “sums of squares” with real analytic coefficients is studied in detail.

MSC: 35H10; 35H20; 35B65

Keywords: Sums of squares; Microlocal Gevrey regularity; Hypoellipticity

1. Introduction

The purpose of this paper is to discuss the Gevrey hypoellipticity properties of three model operators that are sums of squares of vector fields in four dimensions. The operators have analytic coefficients and verify the Hörmander condition: the Lie algebra generated by the vector fields as well as by their commutators has, in every point, dimension equal to the dimension of the ambient space. Hence in view of the celebrated Hörmander theorem, [10], the operators are C^∞-hypoelliptic.

Let $P(x; D) = \sum_1^k X_j^2(x, D)$, $X_j(x, D)$ vector fields with real analytic coefficients on Ω open subset $\mathbb{R}^n$. We say that P is C^∞ (G^r)-hypoelliptic, $r \geq 1$, in Ω if for every U open subset of Ω and every $u \in \mathscr{D}'(U)$, $Pu \in C^\infty(U)$ $(G^r(U))$ implies $u \in C^\infty(U)$ $(G^r(U))$. When $r = 1$ we say

* Address for correspondence: Oskar–Morgenstern–Platz 1, 1090 Vienna, Austria.
E-mail address: gregorio.chinni@gmail.com.

https://doi.org/10.1016/j.jde.2018.03.018
0022-0396/

that P is analytic hypoelliptic. We recall that $G^r(U)$ denotes the r-Gevrey class of function on U: a C^∞-function, f, on U belongs to $G^r(U)$, $1 \leq r \leq \infty$, if for every K compact subset of U there is a constant C_K such that $|\partial^\alpha f(x)| \leq C_K^{|\alpha|+1}(\alpha!)^r$ for every $\alpha \in \mathbb{N}^n$ and $x \in K$.

Derridj showed in [8] that for P as above, the Hörmander condition is necessary for the analytic hypoellipticity but it is not sufficient. An example of operator sum of squares of real analytic vector fields satisfying the Hörmander condition but not analytic hypoelliptic was given by Baouendi and Goulaouic in [3]. At the present, there aren't general analytic hypoellipticity results. Some results, in this direction, were obtained by Treves, [14], Tartakoff, [13], and Albano and Bove, [1]. For completeness, we recall that, with regard to the Gevrey regularity, if no additional assumption is made on the operator P, the (local) optimal characterization was obtained by Derridj and Zuily, [9]. In 1999 Treves formulated a conjecture which related the analytic hypoellipticity with geometrical properties of the characteristic variety of P, see [15] and [16].

In recent papers Albano, Bove and Mughetti, [2], and Bove and Mughetti, [4], showed that the sufficient part of the Treves' conjecture does not hold neither locally nor microlocally. More precisely in [2] and [4] the authors produced and studied the first models which are not consistent with the Treves conjecture, [16]. However, contrary to the cases of [2] and [4], the operators studied here have no exceptional strata because the symbols do not depend on the tangent variables of the "inner most" stratum.

Our results can be stated as follows:

Theorem 1.1. *Let $P_1(x, D)$ the sum of squares given by*

$$D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2 + x_1^{2(r-1)} x_2^{2k} D_4^2 + x_1^{2(r+\ell-1)} D_4^2, \tag{1.1}$$

where p, q, r, k and ℓ are positive integers such that $p < q < r$ and $pk < \ell$ and $P_2(x, D)$ the sum of squares given by

$$D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2 + x_1^{2(r-1)} x_3^{2k} D_4^2 + x_1^{2(r+\ell-1)} D_4^2, \tag{1.2}$$

where p, q, r, k and ℓ are positive integers such that $p < q < r$ and $qk < \ell$. We have:

- **i)** *$P_1(x, D)$ is G^s-hypoelliptic with $s = \sup\left\{\frac{r+kp}{q}, \frac{r}{p}\right\}$.*
- **ii)** *$P_2(x, D)$ is G^s-hypoelliptic with $s = \frac{r+kq}{p}$.*

The strategy used to obtain the above results shows, without particular technical trouble, that:

Remark 1.1. If $pk \geq \ell$ then P_1 is G^s-hypoelliptic with $s = \sup\left\{\frac{r+\ell}{q}, \frac{r}{p}\right\}$ and P_2 is G^s-hypoelliptic with $s = \frac{r+\ell}{p}$.

We recall that by the result of Derridj and Zuily, [9], P_1 is $(r + kp)$-Gevrey hypoelliptic and P_2 is $(r + kq)$-Gevrey hypoelliptic when $kq < \ell$ and they are both $(r + \ell)$-Gevrey hypoelliptic when $kp \geq \ell$.

Theorem 1.2. *Let the operator $P_3(x, D)$ be given by*

$$D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2 + x_1^{2(r-1)} x_2^{2k} D_4^2 + x_1^{2(f-1)} x_3^{2\ell} D_4^2 + x_1^{2(f+e-1)} D_4^2 \tag{1.3}$$