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From second grade fluids to the Navier–Stokes equations

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Abstract

We consider the limit $\alpha \to 0$ for a second grade fluid on a bounded domain with Dirichlet boundary conditions. We show convergence towards a solution of the Navier–Stokes equations under two different types of hypothesis on the initial velocity u_0. If the product $\|u_0\|_{L^2}\|u_0\|_{H^1}$ is sufficiently small we prove global-in-time convergence. If there is no smallness assumption we obtain local-in-time convergence up to the time $C/\|u_0\|^4_{H^1}$.

1. Introduction

We consider in this paper the incompressible second grade fluid equations:

$$\partial_t(u-\alpha\Delta u)-\nu\Delta u+u\cdot\nabla(u-\alpha\Delta u)+\sum_j(u-\alpha\Delta u)_j\nabla u_j=-\nabla p, \qquad \operatorname{div} u=0, \tag{1}$$

where α and ν are some non-negative constants. The fluid is assumed to be enclosed in a bounded region Ω of $\mathbb{R}^3$ and the homogeneous Dirichlet boundary conditions are imposed

$$u(t,\cdot)\big|_{\partial\Omega}=0 \qquad \text{for all } t\geq 0. \tag{2}$$

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The initial value problem is considered and we denote by u_0 the initial velocity:

$$u(0,x)=u_0(x). \tag{3}$$

We assume for simplicity that Ω is C^∞ (although less regularity is actually needed) and that there is no forcing term in (1). It would be possible to include a non-vanishing forcing in our results at the cost of some technical modifications of our proofs; we prefer to avoid these complications in order to keep the paper as simple as possible.

The equations (1) were deduced in [12] from physical principles. Let us just mention here that the second grade fluids are characterized by the following fact: the stress tensor is a polynomial of degree two in the first two Rivlin–Ericksen tensors which are the deformation tensor D and the tensor $(\partial_t + u\cdot\nabla)D$. The vanishing viscosity case $\nu=0$ is also known under the name α-Euler or Euler-α equations and was later obtained via an averaging procedure performed on the classical incompressible Euler equations.

Two main boundary conditions were used for (1) in the mathematical literature: the no-slip boundary conditions and the frictionless slip Navier boundary conditions where the fluid is allowed to slip on the boundary without friction. The second boundary condition is more complex but allows for better mathematical results; also it has less physical relevance. The classical well-posedness results for (1) are the following (see [7–9,15,16] for the Dirichlet boundary conditions and [6] for the Navier boundary conditions):

- In dimension two there exists a unique global H^3 solution if $u_0 \in H^3$.
- In dimension three there exists a unique local H^3 solution if $u_0 \in H^3$. The solution is global if u_0 is small in H^3.

We call H^3 solution a divergence free vector field verifying the boundary conditions and the PDE (1) and who is bounded in time (up to time $t=0$) with values in $H^3(\Omega)$. Let us also mention the paper [2] where solutions in $W^{2,p}$, $p>3$, are constructed.

Let us observe that when $\alpha=0$ relation (1) becomes the Navier–Stokes equations

$$\partial_t u - \nu\Delta u + u\cdot\nabla u = -\nabla p, \qquad \operatorname{div} u = 0 \tag{4}$$

and when $\alpha=\nu=0$ it becomes the Euler equations. It is interesting to know if the solutions of (1) converge to the solutions of the limit equation when $\alpha\to 0$ and $\nu>0$ is fixed or when $\alpha,\nu\to 0$. This was already studied in several papers as we shall see below.

Let us first mention that in the absence of boundaries one can obtain H^3 estimates uniform in α and ν in both dimensions two and three and pass to the limit. This was performed in [18], see also [4] for a simpler proof. But such a result cannot hold true on domains with boundaries. Indeed, if the solutions of (1)–(2) are bounded in H^3 uniformly in α then one can easily pass to the limit $\alpha\to 0$ and obtain at the limit a solution of the Navier–Stokes equations which must also be bounded in H^3. For such a solution to exist, the initial data must verify a compatibility condition. Indeed, one can apply the Leray projector to (4) to obtain that $\partial_t u - \nu\mathbb{P}\Delta u + \mathbb{P}(u\cdot\nabla u) = 0$. Since u vanishes at the boundary, so does $\partial_t u$. We infer that $-\nu\mathbb{P}\Delta u + \mathbb{P}(u\cdot\nabla u)=0$ at the boundary. Observe that u being in H^3 implies that these two terms are in H^1 so the trace at the boundary makes sense. By time continuity we infer that the initial data must verify the compatibility condition $\nu\mathbb{P}\Delta u_0 = \mathbb{P}(u_0\cdot\nabla u_0)$ at the boundary. This is of course in general not verified if we only assume that $u_0\in H^3$ is divergence free and vanishing on the boundary.

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