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Spreading and vanishing in a free boundary problem for nonlinear diffusion equations with a given forced moving boundary ☆

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Abstract

We will study a free boundary problem of the nonlinear diffusion equations of the form $u_t = u_{xx} + f(u)$, $t > 0$, $ct < x < h(t)$, where f is C^1 function satisfying $f(0) = 0$, $c > 0$ is a given constant and $h(t)$ is a free boundary which is determined by a Stefan-like condition. This model may be used to describe the spreading of a new or invasive species with population density $u(t, x)$ over a one dimensional habitat. The free boundary $x = h(t)$ represents the spreading front. In this model, we impose zero Dirichlet boundary condition at left moving boundary $x = ct$. This means that the left boundary of the habitat is a very hostile environment for the species and that the habitat is eroded away by the left moving boundary at constant speed c.

In this paper we will extend the results of a trichotomy result obtained in [23] to general monostable, bistable and combustion types of nonlinearities. We show that the long-time dynamical behavior of solutions can be expressed by unified fashion, that is, for any initial data, the unique solution exhibits exactly one of the behaviors, spreading, vanishing and transition. We also give the asymptotic profile of the solution over the whole domain when spreading happens. The approach here is quite different from that used in [23].

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1. Introduction and main results

The spreading of non-native species has been attracting much interests in population ecology and mathematical modeling on such phenomena has been developed by a lot of researchers since Skellam's work [25]. Recently a new mathematical model was proposed by Du and Lin [6], which is given by

$$\begin{cases} u_t = du_{xx} + u(a - bu), & t > 0,\ 0 < x < h(t), \\ u_x(t,0) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0. \end{cases}$$

The constants a, b, d and μ here are positive and $u_0 \in C^2([0,h_0])$ satisfies $u_0'(0) = u_0(h_0) = 0$, $u_0 > 0$ in $[0,h_0)$. For the problem the existence and uniqueness of global-in-time solutions has been proved. Moreover the asymptotic behavior of solutions has been obtained in the sense that, as $t \to \infty$, either spreading ($h(t) \to \infty$ and $u(t,x) \to a/b$ for each x in $[0,\infty)$) or vanishing ($h(t) \to (\pi/2)\sqrt{d/a}$ and $u(t,\cdot) \to 0$ uniformly in x) occurs. It may help us understand the invasion process by studying, in various and more realistic settings, such a free boundary problem with the Stefan-like condition $h'(t) = -\mu u_x(t,h(t))$. For the first step to this direction we will investigate the effect of erosion by a hostile environment that can cause vanishing in a finite time; the problem is given by

$$\begin{cases} u_t = u_{xx} + f(u), & t > 0,\ ct < x < h(t), \\ u(t,ct) = u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.1}$$

where c, μ and h_0 are given positive constants, so $x = ct$ is a given forced moving boundary with its speed c. Moving boundary $x = h(t)$ is to be determined together with $u(t,x)$. For any given $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, we call a pair $(u(t,x), h(t))$ a classical solution of (1.1) on a time interval $[0,T]$ for some $T > 0$ if it satisfies $u \in C^{1,2}(G_T)$, $h \in C^1([0,T])$ and all the identities in (1.1) are satisfied pointwise, where

$$G_T := \{(t,x) :\ t \in (0,T],\ x \in [ct, h(t)]\}.$$

In this paper we assume for nonlinearity f that $f \in C^1$ and

$$f(0) = 0,\ f(1) = 0,\ f'(1) < 0,\ \ f(u) < 0 \ \text{for}\ u > 1.$$

For given $h_0 > 0$, we assume that initial function u_0 belongs to $\mathscr{X}(h_0)$, where