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Cauchy problems for Keller–Segel type time–space fractional diffusion equation

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Abstract

This paper investigates Cauchy problems for nonlinear fractional time–space generalized Keller–Segel equation ${}_0^cD_t^\beta\rho + (-\triangle)^{\frac{\alpha}{2}}\rho + \nabla\cdot(\rho B(\rho)) = 0$, where Caputo derivative ${}_0^cD_t^\beta\rho$ models memory effects in time, fractional Laplacian $(-\triangle)^{\frac{\alpha}{2}}\rho$ represents Lévy diffusion and $B(\rho) = -s_{n,\gamma}\int_{R^n}\frac{x-y}{|x-y|^{n-\gamma+2}}\rho(y)dy$ is the Riesz potential with a singular kernel which takes into account the long rang interaction. We first establish $L^r - L^q$ estimates and weighted estimates of the fundamental solutions $(P(x,t), Y(x,t))$ (or equivalently, the solution operators $(S_\alpha^\beta(t), T_\alpha^\beta(t))$). Then, we prove the existence and uniqueness of the mild solutions when initial data are in L^p spaces, or the weighted spaces. Similar to Keller–Segel equations, if the initial data are small in critical space $L^{p_c}(\mathbb{R}^n)$ ($p_c = \frac{n}{\alpha+\gamma-2}$), we construct the global existence. Furthermore, we prove the L^1 integrability and integral preservation when the initial data are in $L^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$ or $L^1(\mathbb{R}^n)\cap L^{p_c}(\mathbb{R}^n)$. Finally, some important properties of the mild solutions including the nonnegativity preservation, mass conservation and blowup behaviors are established.

MSC: 26A33; 35A05; 35A08

Keywords: Time–space fractional diffusion equation; Mild solution; Existence and uniqueness; Nonnegativity; Mass conservation; Finite time blow up

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1. Introduction

Fractional derivatives [41,28,14,20,33] are employed to describe the nonlocal effects in time and space. They are integro-differential operators generalizing the definition of integer order derivative to fractional orders, and have been used to deal with numerous application in areas such as physics, hydrology, biomedical engineering, control theory, to name a few [39,30,36,13,49,42,35]. Time fractional derivatives [11,41,28,33] are usually applied to model the ubiquitous memory effects. The theory of time fractional differential equations, especially time fractional ODEs, has been developed by many authors [21,14,17,34]. Both time and spatial fractional derivatives [39,47,13] can be used for anomalous diffusion or dispersion when a particle plume spreads at a rate inconsistent with the Brownian motion models. The appearance of spatial fractional derivatives in diffusion equations are exploited for macroscopic description of transport and often lead to superdiffusion phenomenon. Time fractional derivatives are usually connected with anomalous subdiffusion, where a cloud of particles spread more slowly than a classical diffusion [37,12], because particle sticking and trapping phenomena ordinarily display power-law behaviors.

There have been a lot of works investigating fractional partial differential equation (see [47, 24,15,31,45,50,27] for example). Huang and Liu [24] studied the uniqueness and stability of nonlocal Keller–Segel equations by considering a self-consistent stochastic process driven by rotationally invariant α-stable Lévy process. Taylor [45] constructed the formulas and estimates for the solution to nonhomogeneous time fractional diffusion equations ${}_0^cD_t^\beta u + Au - q(t) = 0$ where A is a positive self-adjoint operator. Zacher [50] considered the regularity of weak solutions to linear diffusion equations with Riemann–Liouville time fractional derivative in a bounded domain in $\mathbb{R}^n$. Kemppainen, Siljander and Zacher [27] performed a careful analysis of the large-time behaviors for fully nonlocal diffusion equations. Allen, Caffarelli and Vasseur [1,2] discussed porous medium flows and parabolic problems with fractional time derivative of Caputo-type.

In this paper, we focus on Cauchy problems of the following nonlinear time–space fractional diffusion equations (NFDE):

$$\begin{cases} {}_0^cD_t^\beta \rho + (-\triangle)^{\frac{\alpha}{2}}\rho + \nabla\cdot(\rho B(\rho)) = 0 \text{ in } (x,t)\in\mathbb{R}^n\times(0,\infty), \\ \rho(x,0) = \rho_0(x), \end{cases} \tag{1.1}$$

where $0<\beta<1$, $1<\alpha\le 2$. ${}_0^cD_t^\beta$ is Caputo fractional derivative operator of order β. Caputo derivative was first introduced in [11] and is more suitable for the initial-value problem compared with the Riemann–Liouville fractional derivative. There are many recent definitions of the Caputo derivative in the literature listed to generalize the traditional definition [20,2,5,33,34]. We will use the definition introduced in [33,34] because of the theoretic convenience (see also Section 2 for the detailed introduction). When the function v is absolutely continuous in time, the definition in [33,34] reduces to the traditional form:

$$ {}_0^cD_t^\beta v(t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\dot{v}(s)ds, \tag{1.2}$$

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