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YJDEQ:9256

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J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Second order smoothness of weak solutions of degenerate linear equations

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Received 22 January 2018

Abstract

Let m and n be positive integers and $\{Y_k(x)\}_{k=1}^m$ be a collection of m first order Lipschitz vector field derivatives in a bounded open set $\Omega \subset \mathbb{R}^n$. Given a real-valued function b in Ω, we find conditions guaranteeing that a weak solution u of $\sum_{k=1}^m Y_k' Y_k u = b$ in Ω has some second order smoothness. More precisely, if Z is a first order vector field derivative, we study when the iterated weak derivatives $ZY_k u$ exist in Ω and belong to $L^2_{loc}(\Omega)$. The main theorem extends a result of the author for the special case when $m = n$ and $Z = Y_i$ for some i. Corollaries include the existence of $ZX_j u$, $j = 1, \dots, n$, for weak solutions u of divergence form equations $\text{div}(Q\nabla u) = b$ if $Q(x)$ is a nonnegative definite, symmetric $n \times n$ matrix and $\{X_j\}_{j=1}^n$ are the derivatives corresponding to the rows of $\sqrt{Q}$. The reverse orders $Y_k Zu$ and $X_j Zu$ are also considered.

MSC: 35J70; 35B65; 35D30

Keywords: Degenerate elliptic partial differential equations; Degenerate quadratic forms; Weak solutions; Second order regularity

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$, m be a positive integer, and $\{W_k(x)\}_{k=1}^m$ be a collection of m Lipschitz continuous vector fields in Ω. Let $u(x) \in L^2(\Omega)$ and suppose that each weak vector

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https://doi.org/10.1016/j.jde.2018.03.012

field derivative $Y_k u = W_k \cdot \nabla u$, $k = 1, \dots, m$, exists in Ω and belongs to $L^2(\Omega)$. We adopt the standard notation

$$H^{1,2}_{\mathcal{Y}}(\Omega) \quad \text{with } \mathcal{Y} = \{Y_k = W_k \cdot \nabla\}_{k=1}^m$$

for the collection of all such u. Here, if W is a Lipschitz vector field on Ω and u is a function in $L^1_{loc}(\Omega)$, we say as usual that the weak vector field derivative $Yu = W \cdot \nabla u$ exists in Ω if there exists $\mu(x) \in L^1_{loc}(\Omega)$ such that for all $\varphi \in Lip_0(\Omega)$,

$$\begin{aligned} \int_\Omega \mu\, \varphi\, dx &= -\int_\Omega u\, Y' \varphi\, dx = -\int_\Omega u\, \mathrm{div}(\varphi W)\, dx \\ &= -\int_\Omega u \left\{ W \cdot \nabla \varphi + \varphi\, \mathrm{div} W \right\} dx. \end{aligned} \tag{1.1}$$

Such μ is clearly unique if it exists, and we then denote $\mu = Yu = W \cdot \nabla u$. Of course, $\nabla \varphi$ exists a.e. in Ω in the usual pointwise sense since $\varphi \in Lip(\Omega)$, but the notation ∇u is an abuse of notation whenever u is a function whose gradient fails to exist in either the pointwise sense or the classical weak sense.

Density of smooth functions will play an important role in our results. By [2], [3] or [5], if $u \in H^{1,2}_{\mathcal{Y}}(\Omega)$, there is a sequence $\{u_\ell\}$ of smooth functions in Ω such that

$$\lim_{\ell \to \infty} \left\{ \|u_\ell - u\|_{L^2(\Omega)} + \|Y_k u_\ell - Y_k u\|_{L^2(\Omega)} \right\} = 0 \quad \text{for every } k. \tag{1.2}$$

We will say that u is represented in $H^{1,2}_{\mathcal{Y}}(\Omega)$ by $\{u_\ell\}$. Conversely, if u is a function in $L^2(\Omega)$ and there is a sequence $\{u_\ell\}$ of smooth functions in Ω (or even just functions in $Lip(\Omega)$) that satisfies (1.2), then $u \in H^{1,2}_{\mathcal{Y}}(\Omega)$, as is easily seen from integration by parts combined with a limit argument. Thus $H^{1,2}_{\mathcal{Y}}(\Omega)$ is characterized by (1.2) just as in the classical result of K. Friedrichs when $m = n$ and the W_k are the standard basis of orthonormal vectors in $\mathbb{R}^n$.

Moreover, if $u \in H^{1,2}_{\mathcal{Y}}(\Omega)$ then for a given open set $\Omega' \subset\subset \Omega$, there are smooth functions u_ℓ defined in Ω' (as opposed to Ω) of convolution form $u_\ell = u * J_{\varepsilon_\ell}$ for a smooth Euclidean approximation of the identity $J_{\varepsilon_\ell}(x) = \varepsilon_\ell^{-n} J(\varepsilon_\ell^{-1} x)$, $\varepsilon_\ell \to 0+$, such that (1.2) holds with Ω' in place of Ω, i.e., such that

$$\lim_{\ell \to \infty} \left\{ \|u_\ell - u\|_{L^2(\Omega')} + \|Y_k u_\ell - Y_k u\|_{L^2(\Omega')} \right\} = 0 \quad \text{for every } k. \tag{1.3}$$

Here, any smooth function $J(x)$ with support in $\{|x| < 1\}$ and $\int_{\mathbb{R}^n} J(x) dx = 1$ can be used, and any sequence $\{\varepsilon_\ell\} \to 0+$ can be used provided $\varepsilon_\ell < dist(\partial\Omega', \partial\Omega)$ for all ℓ.

Reference [3] proves similar density results for higher order iterates of vector field derivatives that will be used below.

It is well known that $H^{1,2}_{\mathcal{Y}}(\Omega)$ is a Banach space with norm

$$\|u\|_{H^{1,2}_{\mathcal{Y}}(\Omega)} = \left(\int_\Omega u^2\, dx + \int_\Omega \sum_{k=1}^m (Y_k u)^2\, dx \right)^{\frac{1}{2}}.$$

Please cite this article in press as: R.L. Wheeden, Second order smoothness of weak solutions of degenerate linear equations, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.03.012