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Minimal random attractors

Hans Crauel [a,*], Michael Scheutzow [b]

[a] Institut für Mathematik, 60629 Frankfurt, FRG

[b] Institut für Mathematik, MA 7-5, Fakultät II, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, FRG

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Abstract

It is well-known that random attractors of a random dynamical system are generally not unique. We show that for general pullback attractors and weak attractors, there is always a minimal (in the sense of smallest) random attractor which attracts a given family of (possibly random) sets. We provide an example which shows that this property need not hold for forward attractors. We point out that our concept of a random attractor is very general: The family of sets which are attracted is allowed to be completely arbitrary.

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1. Introduction

For deterministic dynamical systems on metric spaces the notion of an attractor is well established. The most common notion, mainly used for partial differential equations (PDEs) on suitable Hilbert or Banach spaces, is that of a *global set attractor*. It is characterized by being a compact set, being strictly invariant, and attracting every compact, or even every bounded set. Uniqueness of the global set attractor is immediate.

* Corresponding author.

E-mail addresses: crauel@math.uni-frankfurt.de (H. Crauel), ms@math.tu-berlin.de (M. Scheutzow).

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Another very common notion is that of a *(global) point attractor*, which is often used for systems on locally compact spaces (which are often Euclidean spaces or finite-dimensional manifolds). A global point attractor is again compact and strictly invariant, but it is only assumed to attract every point (or, equivalently, every finite set). The global point attractor is in general not unique, which can be seen from the simple dynamical system induced by the scalar differential equation $\dot{x} = x - x^3$. Here the unique global set attractor is the interval $[-1, 1]$, which, of course, is also a point attractor. But also $[-1, 0] \cup \{1\}$, and $\{-1\} \cup [0, 1]$, are point attractors. Obviously, there is a minimal point attractor, namely $\{-1, 0, 1\}$.

In fact, for deterministic systems the following result is well known: Whenever $\mathcal{B}$ is an arbitrary family of non-empty subsets of the state space then there exists an attractor for $\mathcal{B}$ (i.e., a compact, strictly invariant set attracting every $B \in \mathcal{B}$) if and only if there exists a compact set such that every $B \in \mathcal{B}$ is attracted by this set. Furthermore, there exists a unique minimal attractor for $\mathcal{B}$, which is given by the closure of the union of the ω-limit sets of all elements of $\mathcal{B}$. This minimal attractor for $\mathcal{B}$ is addressed as *the* $\mathcal{B}$-attractor. If a global set attractor exists then whenever a $\mathcal{B}$-attractor exists it is always a subset of the global set attractor.

For random dynamical systems an analogous statement has been established in [6]. However, this result had to assume a separability condition for $\mathcal{B}$.

The aim of the present paper is to remove this separability condition, i.e. to establish that for any family $\mathcal{B}$ of (possibly even random) sets for which there is a compact random set attracting every element of $\mathcal{B}$, there exists a unique minimal random attractor for $\mathcal{B}$.

Furthermore, it is in general not true that this $\mathcal{B}$-attractor is given by the closed random set

$$\overline{\bigcup_{B \in \mathcal{B}} \Omega_B(\omega)} \quad \text{almost surely,} \tag{1}$$

where $\Omega_B(\omega)$ denotes the (random) Ω-limit set of B. This is shown using an example of a random dynamical system induced by a stochastic differential equation on the unit circle S^1. Here the global set attractor is the whole S^1 (which is a strictly invariant compact set). If $\mathcal{B}$ is taken to be the family of all deterministic points in S^1 it is shown that (1) gives S^1, the global set attractor, almost surely. However, the minimal point attractor is a one point set consisting of a random variable, which (pullback) attracts every solution starting in a deterministic point.

2. Notation and preliminaries

Let E be a Polish space, i.e. a separable topological space whose topology is metrizable with a complete metric. Several assertions in the following are formulated in terms of a metric d on E which is referred to without further mentioning. This metric will always be assumed to generate the topology of E and to be complete, even if some of the assertions hold also if d is not complete. For $x \in E$ and $A \subset E$ we define $d(x, A) = \inf\{d(x, a) : a \in A\}$ with the convention $d(x, \varnothing) = \infty$. For non-empty subsets A and B of E we denote the Hausdorff semi-metric by $d(B, A) := \sup\{d(b, A) : b \in B\}$ and define $d(\varnothing, A) := 0$ and $d(B, \varnothing) := \infty$ in case $B \neq \varnothing$.

We note that with this convention both for the empty family $\mathcal{B} = \varnothing$ as well as for $\mathcal{B} = \{\varnothing\}$ the empty set $A = \varnothing$ is an attractor, in fact the minimal one.

We denote the Borel σ-algebra on E (i.e. the smallest σ-algebra on E which contains every open set) by $\mathcal{E}$.