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On the existence of solutions to a one-dimensional degenerate nonlinear wave equation

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Abstract

This paper is concerned with the degenerate initial–boundary value problem to the one-dimensional nonlinear wave equation $u_{tt} = ((1+u)^a u_x)_x$ which arises in a number of various physical contexts. The global existence of smooth solutions to the degenerate problem was established under relaxed conditions on the initial–boundary data by the characteristic decomposition method. Moreover, we show that the solution is uniformly $C^{1,\alpha}$ continuous up to the degenerate boundary and the degenerate curve is $C^{1,\alpha}$ continuous for $\alpha \in (0, \min\{\frac{a}{1+a}, \frac{1}{1+a}\})$.

MSC: 35L20; 35L70; 35L80

Keywords: Nonlinear wave equation; Degenerate hyperbolic; Characteristic decomposition; Bootstrap

1. Introduction

In this paper, we are focused on the following one-dimensional nonlinear wave equation

$$u_{tt} - ((1+u)^{2a} u_x)_x = 0, \tag{1.1}$$

where (t, x) are the time–space independent variables, $u = u(t, x)$ is the dependent variable and $a > 0$ is a constant. Equation (1.1) can be used a model to describe the flow of one-dimensional

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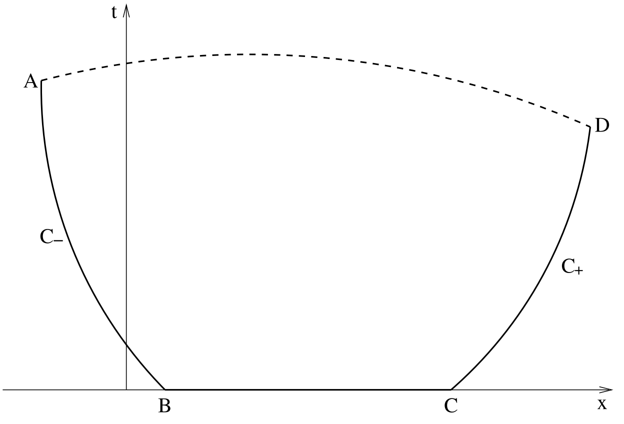

Fig. 1. The domain $ABCD$.

gas, shallow water waves theory, longitudinal wave propagation on a moving threadline, dynamics of a finite nonlinear string, elastic-plastic materials, and electromagnetic transmission line (see pp. 50–52 in [1]). This equation and its corresponding conservation system have been widely studied by many authors, see e.g. [2,3,7,9,15,16].

Part of the motivation of this paper comes from a recent work by Sugiyama [19]. In [19], Sugiyama investigated the large time behavior of solutions of (1.1) and obtained a sufficient condition for the occurrence of the degeneracy of the equation in finite time. More precisely, the author proved that, under the relaxed conditions and an additional condition for the initial data (i.e., $\int_{\mathbb{R}} u_t(0,x)\mathrm{d}x < -2/(a+1)$), there exists $T^* > 0$ such that the Cauchy problem of (1.1) exists a smooth solution u in $[0, T^*) \times \mathbb{R}$ and the function $1 + u(t, x_0)$ approaches zero as $t \nearrow T^*$ for some $x_0 \in \mathbb{R}$. Moreover, he also pointed out that if $\int_{\mathbb{R}} u_t(0,x)\mathrm{d}x > -2/(a+1)$ and the initial data satisfying the relaxed conditions, then the degeneracy does not occur and the Cauchy problem of (1.1) exists a global smooth solution in $\mathbb{R}^+ \times \mathbb{R}$.

In the present paper, we consider equation (1.1) in a completely different perspective compared to the research of Sugiyama [19]. Specifically, we specify the boundary data on two boundary curves such that the equation degenerates at the endpoints of these boundary curves and explore the existence of smooth solutions to its degenerate initial–boundary value problem and the regularity of the degenerate curves. To the best of our knowledge, until recently, studies on the degenerate hyperbolic problems for the nonlinear wave equations are still very limited. We expect this work can help unveiling the structure of solutions near a degenerate curve for nonlinear wave equation (1.1) and the related equations.

1.1. The problem and the main result

We consider in this paper the following degenerate hyperbolic problem

Problem 1. Let B and C be two points on the x-axis with $x_B < x_C$. Given initial data on segment BC such that $u > -1$ and given a negative characteristic $\widehat{BA}$ and a positive characteristic $\widehat{CD}$ such that $u = -1$ at points A and D. We look for solutions to (1.1) in the domain $ABCD$ with the data on $\widehat{ABCD}$, see Fig. 1.

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