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The damped wave equation with unbounded damping ☆

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Abstract

We analyze new phenomena arising in linear damped wave equations on unbounded domains when the damping is allowed to become unbounded at infinity. We prove the generation of a contraction semigroup, study the relation between the spectra of the semigroup generator and the associated quadratic operator function, the convergence of non-real eigenvalues in the asymptotic regime of diverging damping on a subdomain, and we investigate the appearance of essential spectrum on the negative real axis. We further show that the presence of the latter prevents exponential estimates for the semigroup and turns out to be a robust effect that cannot be easily canceled by adding a positive potential. These analytic results are illustrated by examples.

MSC: 35L05; 35P05; 47A56; 47D06

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1. Introduction

We consider the spectral problem associated with the linearly damped wave equation

$$u_{tt}(t,x) + 2a(x)u_t(t,x) = (\Delta - q(x))u(t,x), \quad t > 0, \quad x \in \Omega, \tag{1.1}$$

with non-negative damping a and potential q on an open (typically unbounded) subset Ω of $\mathbb{R}^d$; when Ω is not all of $\mathbb{R}^d$ we shall impose Dirichlet boundary conditions on its boundary $\partial\Omega$. Here both the potential q and the damping a are allowed to be unbounded and/or singular.

The main goal of this paper is to analyze the new phenomena which arise when the damping term a is allowed to grow to infinity on an unbounded domain. To this end, we formally rewrite (1.1) as a first order system

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & I \\ \Delta - q & -2a \end{pmatrix}}_{G} \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.2}$$

and realize the operator G in a suitable Hilbert space without assuming that the damping is dominated by $\Delta - q$. Our main results show that, even under these weak assumptions on the damping, G generates a contraction semi-group, but G may have essential spectrum that covers the entire semi-axis $(-\infty, 0]$. As a consequence, although the energy of solutions will still approach zero, this decay will now be polynomial and no longer exponential, *cf.* [11] and the discussion below. We further establish conditions for the latter and study the convergence of non-real eigenvalues in the asymptotic regime of diverging damping on a subdomain.

In most of the literature on linearly damped wave equations on unbounded domains only bounded damping terms were considered. This is a natural condition to allow for the exponential decay of solutions, while large damping terms in fact tend to weaken the decay giving rise to the phenomenon known as over-damping. More precisely, increasing the damping term past a certain threshold will cause part of the spectrum to approach the imaginary axis, thus producing a slower decay. This phenomenon may already occur in finite-dimensional systems and in equations like (1.1) with bounded damping where its effect on individual eigenvalues is well-understood. Unbounded accretive or sectorial damping terms of equal strength as $-\Delta$ were considered as an application of semigroup generation results and of spectral estimates for second order abstract Cauchy problems in [13,12] which allow to control the spectrum, in particular, near the imaginary axis.

To the best of our knowledge, the only article where the damping has been allowed to become unbounded at infinity is the recent preprint [11]. There the authors consider dampings on all of $\mathbb{R}^d$ ($d \geq 3$) and, using methods different from ours, they prove the existence and uniqueness of weak solutions whose energy decays at least with $(1+t)^{-2}$. In fact, our result on the essential spectrum will show that one of the characteristics of such systems is that the essential spectrum covers the whole semi-axis $(-\infty, 0]$, thus excluding exponential energy decay.

To illustrate this issue, consider the simple model case given by the generators G_n, $n \in \mathbb{N}$, of the wave equation on the real line with the family of damping terms

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