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Differential Galois theory and non-integrability of planar polynomial vector fields

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Abstract

We study a necessary condition for the integrability of the polynomials vector fields in the plane by means of the differential Galois Theory. More concretely, by means of the variational equations around a particular solution it is obtained a necessary condition for the existence of a rational first integral. The method is systematic starting with the first order variational equation. We illustrate this result with several families of examples. A key point is to check whether a suitable primitive is elementary or not. Using a theorem by Liouville, the problem is equivalent to the existence of a rational solution of a certain first order linear equation, the Risch equation. This is a classical problem studied by Risch in 1969, and the solution is given by the "Risch algorithm". In this way we point out the connection of the non integrability with some higher transcendent functions, like the error function.

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1. Introduction

The problem of the *integrability* of planar vector fields has attracted the attention of many mathematicians during decades. Among different approaches, *Galois Theory* of linear differential equations has played an important rôle in its understanding, even in the *a priori* (so far) simpler case of polynomial vector fields (see [4,24,1] and references therein). For instance, the application of differential Galois Theory to variational equations along a given integral curve constitutes a powerful criterion of non-integrability for Hamiltonian systems (see [16]). Extensions of this method for some non Hamiltonian vector fields have been carried out by Ayoul and Zung [2]. They strongly rely on the main result of [18].

Many authors have tackled the problem of integrability from other different points of view. For instance, we refer the reader to the papers [19,23,27,28] and references therein, where necessary conditions on Liouvillian and Elementary integrability are provided.

The aim of this paper is to apply Galois Theory to prove the non-existence of rational first integrals for a kind of planar polynomial vector fields. Indeed, let us consider planar vector fields of the form

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}, \tag{1}$$

with P, Q analytic functions in some domain of $\mathbb{C}^2$ and assume $\Gamma : y - \varphi(x) = 0$ to be an integral curve[1] of X. This is equivalent to say that Γ is a solution (a *leaf*) of the first order differential equation

$$y' = \frac{Q}{P} = f(x, y), \tag{2}$$

which defines its associated foliation (orbits of the vector field X). From now on $'$ will denote derivative with respect to the spatial variable x. The behaviour around the solution Γ is usually approached by studying its variational equations. In our case, with respect to equation (2). Precisely, let $\phi(x, y)$ denote the flow of (2). Consider (x_0, y_0) a point in Γ, that is, $y_0 = \varphi(x_0)$. Note that (x_0, y_0) is the initial condition defining Γ: $\varphi(x) = \phi(x, y_0)$. We are interested in the variation of the flow ϕ respect to the initial condition y, around $y = y_0$ and keeping $x = x_0$ fixed. In other words, in the flow defined by the initial condition $\phi(x_0, y) = y$. This means that we want to compute the Taylor expansion coefficients

$$\varphi_k(x) = \frac{\partial^k \phi}{\partial y^k}(x, y_0)$$

for which

$$\phi(x, y) = \varphi(x) + \frac{\partial \phi}{\partial y}(x, y_0)(y - y_0) + \frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}(x, y_0)(y - y_0)^2 + \cdots,$$

[1] Usually it is also referred as an *orbit* of the ode system $\dot{x} = P$, $\dot{y} = Q$. In qualitative theory of dynamical systems this is commonly called *invariant* curve.