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YJDEQ:9223

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J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

A proof of Wright's conjecture

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Received 31 March 2017; revised 10 January 2018

Abstract

Wright's conjecture states that the origin is the global attractor for the delay differential equation $y'(t) = -\alpha y(t-1)[1 + y(t)]$ for all $\alpha \in (0, \frac{\pi}{2}]$ when $y(t) > -1$. This has been proven to be true for a subset of parameter values α. We extend the result to the full parameter range $\alpha \in (0, \frac{\pi}{2}]$, and thus prove Wright's conjecture to be true. Our approach relies on a careful investigation of the neighborhood of the Hopf bifurcation occurring at $\alpha = \frac{\pi}{2}$. This analysis fills the gap left by complementary work on Wright's conjecture, which covers parameter values further away from the bifurcation point. Furthermore, we show that the branch of (slowly oscillating) periodic orbits originating from this Hopf bifurcation does not have any subsequent bifurcations (and in particular no folds) for $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + 6.830 \times 10^{-3}]$. When combined with other results, this proves that the branch of slowly oscillating solutions that originates from the Hopf bifurcation at $\alpha = \frac{\pi}{2}$ is globally parametrized by $\alpha > \frac{\pi}{2}$.

Keywords: Delay differential equation; Hopf bifurcation; Wright's conjecture; Supercritical bifurcation branch; Newton–Kantorovich theorem

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1 Partially supported by NWO VICI-grant 639.033.109.

2 Partially supported by NSF DMS 0915019, NSF DMS 1248071.

https://doi.org/10.1016/j.jde.2018.02.018

1. Introduction

In many biological and physical systems the dependency of future states relies not only on the present situation, but on a broader history of the system. For simplicity, mathematical models often ignore the causal influence of all but the present state. However, in a wide variety of applications delayed feedback loops play an inextricable role in the qualitative dynamics of a system [13]. These phenomena can be modeled using delay and integro-differential equations, the theory of which has developed significantly over the past 60 years [7]. A canonical and well-studied example of a nonlinear delay differential equation is Wright's equation:

$$y'(t) = -\alpha\, y(t-1)\,[1 + y(t)]\,. \tag{1.1}$$

Here α is considered to be both real and positive. This equation has been a central example considered in the development of much of the theory of functional differential equations. For a short overview of this equation, we refer the reader to [6]. We cite some basic properties of its global dynamics [24]:

- Corresponding to every $y \in C^0([-1, 0])$, there is a unique solution of (1.1) for all $t > 0$.
- Wright's equation has two equilibria $y \equiv -1$ and $y \equiv 0$. Moreover, solutions cannot cross -1. Any solution with $y(t_0) = -1$ (for some $t_0 \geq 0$) is identically equal to -1 for $t \geq 0$.
- When $y(0) < -1$ then the solution decreases monotonically (for $t > 1$) without bound.
- When $y(0) > -1$ then $y(t)$ is globally bounded as $t \to +\infty$.

Henceforth we restrict our attention to $y(t) > -1$. In Wright's seminal 1955 paper [24], he showed that if $\alpha \leq \frac{3}{2}$, then any solution having $y(t) > -1$ is attracted to 0 as $t \to +\infty$. Wright suggested that $y \equiv 0$ could be the global attractor for a larger range of α. The natural upper limit for this range is $\alpha = \frac{\pi}{2}$, where the equilibrium $y \equiv 0$ changes from asymptotically stable to unstable, a claim which has come to be known as Wright's conjecture:

Conjecture 1.1 *(Wright's conjecture). For every $0 < \alpha \leq \frac{\pi}{2}$, the zero solution to (1.1) is globally attractive.*

For $\alpha > \frac{\pi}{2}$, Wright proved the existence of oscillatory solutions to (1.1) which do not tend towards 0, and whose zeros are spaced at distances greater than the delay. Such a periodic solution is said to be *slowly oscillating*, and formally defined as follows:

Definition 1.2. A *slowly oscillating periodic solution (SOPS)* is a periodic solution $y(t)$ which up to a time translation satisfies the following property: there exists some $t_-, t_+ > 1$ and $L = t_- + t_+$ such that $y(t) > 0$ for $t \in (0, t_+)$, $y(t) < 0$ for $t \in (-t_-, 0)$, and $y(t + L) = y(t)$ for all t, so that L is the minimal period of $y(t)$.

In Jones' 1962 paper [11] he proved that for $\alpha > \frac{\pi}{2}$ there exists a slowly oscillating periodic solution to (1.1). Based on numerical calculations [12] Jones made the following conjecture:

Conjecture 1.3 *(Jones' conjecture). For every $\alpha > \frac{\pi}{2}$ there exists a unique (up to time translation) slowly oscillating periodic solution to (1.1).*

Please cite this article in press as: J.B. van den Berg, J. Jaquette, A proof of Wright's conjecture, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.02.018