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A criticality result for polycycles in a family of quadratic reversible centers ☆

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Abstract

We consider the family of dehomogenized Loud's centers $X_\mu = y(x-1)\partial_x + (x + Dx^2 + Fy^2)\partial_y$, where $\mu = (D, F) \in \mathbb{R}^2$, and we study the number of critical periodic orbits that emerge or disappear from the polycycle at the boundary of the period annulus. This number is defined exactly the same way as the well-known notion of cyclicity of a limit periodic set and we call it criticality. The previous results on the issue for the family $\{X_\mu, \mu \in \mathbb{R}^2\}$ distinguish between parameters with criticality equal to zero (regular parameters) and those with criticality greater than zero (bifurcation parameters). A challenging problem not tackled so far is the computation of the criticality of the bifurcation parameters, which form a set Γ_B of codimension 1 in $\mathbb{R}^2$. In the present paper we succeed in proving that a subset of Γ_B has criticality equal to one.

MSC: 34C07; 34C23; 34C25

Keywords: Center; Period function; Critical periodic orbit; Bifurcation; Criticality

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1. Introduction and statement of the result

In the present paper we study the local bifurcation diagram of the period function associated to a family of quadratic centers. By local we mean near the polycycle at the boundary of the period annulus of the center. In the literature one can find different terminology to classify the quadratic centers but essentially there are four families: Hamiltonian, reversible Q_3^R, codimension four Q_4 and generalized Lotka–Volterra systems Q_3^{LV}. According to Chicone's conjecture [2], the reversible centers have at most two critical periodic orbits, whereas the centers of the other three families have monotonic period function. In this context critical periodic orbits play exactly the same role as limit cycles in the celebrated Hilbert's 16th problem (see for instance [6] and references therein). What is more, from the point of view of the techniques, results and notions involved, Chicone's conjecture is the counterpart for the period function to the question of whether quadratic polynomial differential systems have at most four limit cycles, i.e., $H(2) = 4$. Both problems are far from being solved and pose challenging difficulties. There are many papers proving partial results related to Chicone's conjecture and there is much analytic evidence that it is true. In this direction, and without being exhaustive, let us quote Coppel and Gavrilov [4], who showed that the period function of any Hamiltonian quadratic center is monotonic, and Zhao [18], who proved the same property for the Q_4 centers. There are very few results concerning the Q_3^{LV} centers. In the middle 80s several authors [7,14,17] showed independently the monotonicity of the classical Lotka–Volterra centers, which constitute a hypersurface inside the Q_3^{LV} family, and more recently the same property has been proved in [16] for two other hypersurfaces. With regard to reversible quadratic centers, it is well known that they can be brought by an affine transformation and a constant rescaling of time to the *Loud normal form*

$$\begin{cases} \dot{x} = -y + Bxy, \\ \dot{y} = x + Dx^2 + Fy^2. \end{cases}$$

It is proved in [5] that if $B = 0$ then the period function of the center at the origin is globally monotone. So, from the point of view of the study of the period function, the most interesting stratum of quadratic centers is $B \neq 0$, which can be brought by means of a rescaling to $B = 1$, i.e.,

$$X_\mu \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2, \end{cases} \tag{1}$$

where $\mu := (D, F) \in \mathbb{R}^2$. This paper is addressed to study the period function of the center at the origin in this two-parametric family. More precisely, for a given $\hat{\mu} \in \mathbb{R}^2$, we are concerned with the number of critical periodic orbits of X_μ with $\mu \approx \hat{\mu}$ that emerge or disappear from the polycycle $\Pi_{\hat{\mu}}$ of $X_{\hat{\mu}}$ at the boundary of its period annulus as we move slightly the parameter. We refer to this number as the *criticality* of the polycycle, $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big)$, see Definition 2.1. (Again this is the counterpart to the notion of cyclicity of a limit periodic set, see for instance [15].) Then we say that $\hat{\mu} \in \mathbb{R}^2$ is a local regular parameter if $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$ and that it is a local bifurcation parameter if $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) \geqslant 1$. The initial work on this issue is [11] and, since our result is closely related to the ones obtained there, next we explain them succinctly. With this aim, let Γ_U be the union of dotted straight lines in Fig. 1, whatever its color is. Consider also the bold curve Γ_B. (Here the subscripts B and U stand for bifurcation and unspecified respectively.) Then, following this notation, [11, Theorem A] shows that the open