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Global low-energy weak solution and large-time behavior for the compressible flow of liquid crystals

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Abstract

In this paper, we consider the weak solution of the simplified Ericksen–Leslie system modeling compressible nematic liquid crystal flows in $\mathbb{R}^3$. When the initial data are of small energy and initial density is positive and essentially bounded, we prove the existence of a global weak solution in $\mathbb{R}^3$. The large-time behavior of a global weak solution is also established.

MSC: 35J20; 35J65

Keywords: Global existence; Weak solution; Large-time behavior; Liquid crystals

1. Introduction

We consider the following hydrodynamic system modeling the flow of nematic liquid crystal materials [2,6,22]:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P(\rho) = \mu \Delta u + \lambda \nabla \mathrm{div} u - \nabla d \Delta d, \\ \partial_t d + u \cdot \nabla d = \Delta d + |\nabla d|^2 d, \end{cases} \tag{1.1}$$

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for $(t, x) \in [0, +\infty) \times \mathbb{R}^3$. Here ρ, $u = (u^1, u^2, u^3)^t$ and P denote the density, the velocity, and the pressure respectively. $d = (d^1, d^2, d^3)^t$ is the unit-vector ($|d| = 1$) on the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ representing the macroscopic molecular orientation of the liquid crystal materials. μ and λ are positive viscosity constants, and div and Δ are the usual spatial divergence and Laplace operators.

The above system (1.1) is a simplified version of the Ericksen–Leslie model for the hydrodynamics of nematic liquid crystals. Roughly speaking, the system (1.1) is a coupling between the non-homogeneous Navier–Stokes equations and the transported flow harmonic maps. Due to the physical importance and mathematical challenges, the study on nematic liquid crystals has attracted many physicists and mathematicians. The mathematical analysis of the incompressible liquid crystal flows was initiated by Lin and Liu in [23,24]. For any bounded smooth domain in $\mathbb{R}^2$, Lin, Lin and Wang [25] have proved the global existence of Leray–Hopf type weak solutions to system (1.1) which are smooth everywhere except on finitely many time slices (see [9] for the whole space). The uniqueness of weak solutions in two dimension was studied by [26,35]. Hong and Xin [10] studied the global existence for general Ericksen–Leslie system in dimension two. In [34], Wang proved the global existence of strong solutions in whole space under some small conditions. Recently, Lin et al. [14,27] obtained the global existence of weak solutions the nematic liquid crystal flow in dimension three under geometric angle condition and constructed the examples of finite time singularity for any generic initial data.

When the fluid is allowed to be compressible, the Ericksen–Leslie system becomes more complicate. To our knowledge, there seems very few analytic works available yet. The local-in-time strong solutions to the initial value or initial boundary value problem of system (1.1) with non-negative initial density were studied in [3,12]. Based on [16,17], the blow up criterion of strong solutions were obtained in [12,13]. The global existence and uniqueness of strong solution in critical space were studied in [11]. Motivated by [15], when the initial data was sufficiently smooth and suitably small in some energy-norm, the global well-posedness of classical solutions were proved in [21]. Especially global weak solutions were established in [18,19,28] under some small condition or geometric angle condition (see [20]).

Our aim in this paper is to establish the global existence of low-energy weak solutions of system (1.1), if the following initial value:

$$(\rho(\cdot, 0), u(\cdot, 0), d(\cdot, 0)) = (\rho_0, u_0, d_0), \tag{1.2}$$

satisfies that ρ_0 is bounded above and below away from zero, $|d_0| = 1$, $u_0, \nabla d_0 \in L^p(\mathbb{R}^3)$ for some $p > 6$ satisfying (1.10) and $(\rho_0, u_0, \nabla d_0)$ is small in $L^2(\mathbb{R}^3)$. Thus the total initial energy is small, but no other smallness or regularity conditions are imposed.

When the direction field d does not appear, (1.1) reduces to the compressible Navier–Stokes equations. The global classical solutions were first obtained by Matsumura–Nishida [30,31] for initial data close to a non-vacuum equilibrium in $H^3(\mathbb{R}^3)$. In particular, the theory requires that the solution has small oscillations from a uniform non-vacuum state so that the density is strictly away from the vacuum and the gradient of the density remains bounded uniformly in time. Later, Hoff [7,8] studied the problem for discontinuous initial data. For the existence of solutions for arbitrary data, the major breakthrough is due to Lions [29] (see also Feireisl et al. [4]), where he obtains global existence of weak solutions-defined as solutions with finite energy. Suen and Hoff [33] adopted Hoff's techniques to obtain global existence of low-energy weak solutions for the magnetohydrodynamics. In this paper, we shall study the Cauchy problem (1.1)–(1.2) for liquid crystals and establish the global existence and large time behavior of low-energy weak solutions. However, compared with the compressible Navier–Stokes equations, some new difficulties arise

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