ARTICLE IN PRESS YJDEQ:9200

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

On the exterior Dirichlet problem for Hessian quotient equations ☆

Dongsheng Li [b], Zhisu Li [a,b,*]

[a] Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China

[b] School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China

Received 5 October 2017

Abstract

In this paper, we establish the existence and uniqueness theorem for solutions of the exterior Dirichlet problem for Hessian quotient equations with prescribed asymptotic behavior at infinity. This extends the previous related results on the Monge–Ampère equations and on the Hessian equations, and rearranges them in a systematic way. Based on the Perron's method, the main ingredient of this paper is to construct some appropriate subsolutions of the Hessian quotient equation, which is realized by introducing some new quantities about the elementary symmetric polynomials and using them to analyze the corresponding ordinary differential equation related to the generalized radially symmetric subsolutions of the original equation.

MSC: 35D40; 35J15; 35J25; 35J60; 35J96

Keywords: Dirichlet problem; Existence and uniqueness; Exterior domain; Hessian quotient equation; Perron's method; Prescribed asymptotic behavior

☆ Research supported by NSFC 11671316.

* Corresponding author.

E-mail addresses: lidsh@mail.xjtu.edu.cn (D.-S. Li), lizhisu@bicmr.pku.edu.cn, lizhisu@stu.xjtu.edu.cn (Z.-S. Li).

https://doi.org/10.1016/j.jde.2018.01.047

0022-0396/© 2018 Published by Elsevier Inc.

Please cite this article in press as: D.-S. Li, Z.-S. Li, On the exterior Dirichlet problem for Hessian quotient equations, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.01.047

1. Introduction

In this paper, we consider the Dirichlet problem for the Hessian quotient equation

$$\frac{\sigma_k(\lambda(D^2u))}{\sigma_l(\lambda(D^2u))} = 1 \tag{1.1}$$

in the exterior domain $\mathbb{R}^n \setminus \overline{D}$, where D is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, $0 \leq l < k \leq n$, $\lambda(D^2u)$ denotes the eigenvalue vector $\lambda := (\lambda_1, \lambda_2, ..., \lambda_n)$ of the Hessian matrix D^2u of the function u, and

$$\sigma_0(\lambda) \equiv 1 \quad \text{and} \quad \sigma_j(\lambda) := \sum_{1 \leq s_1 < s_2 < ... < s_j \leq n} \lambda_{s_1}\lambda_{s_2}...\lambda_{s_j} \ (\forall 1 \leq j \leq n)$$

are the elementary symmetric polynomials of the n-vector λ. Note that when $l = 0$, (1.1) is the Hessian equation $\sigma_k(\lambda(D^2u)) = 1$; when $l = 0$, $k = 1$, it is the Poisson equation $\Delta u = 1$, a linear elliptic equation; when $l = 0$, $k = n$, it is the famous Monge–Ampère equation $\det(D^2u) = 1$; and when $l = 1$, $k = 3$, $n = 3$ or 4, it is the special Lagrangian equation $\sigma_1(\lambda(D^2u)) = \sigma_3(\lambda(D^2u))$ in three or four dimension (in three dimension, this is $\det(D^2u) = \Delta u$ indeed) which arises from the special Lagrangian geometry [17].

For linear elliptic equations of second order, there have been much extensive studies on the exterior Dirichlet problem, see [28] and the references therein. For the Monge–Ampère equation, a classical theorem of Jörgens [22], Calabi [3] and Pogorelov [29] states that any convex classical solution of $\det(D^2u) = 1$ in $\mathbb{R}^n$ must be a quadratic polynomial. Related results was also given by [10], [4], [32] and [23]. Caffarelli and Li [7] extended the Jörgens–Calabi–Pogorelov theorem to exterior domains. They proved that if u is a convex viscosity solution of $\det(D^2u) = 1$ in the exterior domain $\mathbb{R}^n \setminus \overline{D}$, where D is a bounded domain in $\mathbb{R}^n$, $n \geq 3$, then there exist $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that

$$\limsup_{|x|\to+\infty} |x|^{n-2} \left| u(x) - \left(\frac{1}{2}x^T Ax + b^T x + c\right)\right| < \infty. \tag{1.2}$$

With such prescribed asymptotic behavior at infinity, they also established an existence and uniqueness theorem for solutions of the Dirichlet problem of the Monge–Ampère equation in the exterior domain of $\mathbb{R}^n$, $n \geq 3$. See [14], [15] or [13] for similar problems in two dimension. Recently, J.-G. Bao, H.-G. Li and Y.-Y. Li [2] extended the above existence and uniqueness theorem of the exterior Dirichlet problem in [7] for the Monge–Ampère equation to the Hessian equation $\sigma_k(\lambda(D^2u)) = 1$ with $2 \leq k \leq n$ and with some appropriate prescribed asymptotic behavior at infinity which is modified from (1.2). Before them, for the special case that $A = c_0 I$ with $c_0 := (C_n^k)^{-1/k}$ and $C_n^k := n!/(k!(n-k)!)$, the exterior Dirichlet problem for the Hessian equation has been investigated by Dai and Bao in [12]. At the same time, Dai [11] proved the existence theorem of the exterior Dirichlet problem for the Hessian quotient equation (1.1) with $k - l \geq 3$, and with the prescribed asymptotic behavior at infinity of the special case that $A = c_* I$, that is,

$$\limsup_{|x|\to+\infty} |x|^{k-l-2} \left| u(x) - \left(\frac{c_*}{2}|x|^2 + c\right)\right| < \infty, \tag{1.3}$$

Please cite this article in press as: D.-S. Li, Z.-S. Li, On the exterior Dirichlet problem for Hessian quotient equations, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.01.047