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Center manifolds for a class of degenerate evolution equations and existence of small-amplitude kinetic shocks ☆

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Abstract

We construct center manifolds for a class of degenerate evolution equations including the steady Boltzmann equation and related kinetic models, establishing in the process existence and behavior of small-amplitude kinetic shock and boundary layers. Notably, for Boltzmann's equation, we show that elements of the center manifold decay in velocity at *near-Maxwellian rate*, in accord with the formal Chapman–Enskog picture of near-equilibrium flow as evolution along the manifold of Maxwellian states, or Grad moment approximation via Hermite polynomials in velocity. Our analysis is from a classical dynamical systems point of view, with a number of interesting modifications to accommodate ill-posedness of the underlying evolution equation.

Keywords: Degenerate evolution equation; Center manifold; Steady Boltzmann equation; Boltzmann shock profile; Boltzmann boundary layer

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1. Introduction

In this paper, we study existence and properties of near-equilibrium steady solutions, including in particular small-amplitude shock and boundary layers, of kinetic-type relaxation systems

$$A^0\mathbf{u}_t + A\mathbf{u}_x = Q(\mathbf{u}), \tag{1.1}$$

on a general Hilbert space $\mathbb{H}$, where A^0, A are given (constant) bounded linear operator and Q is a bounded bilinear map (cf. [23,26]). More generally, we study existence and approximation of center manifolds for a class of degenerate evolution equations arising as steady equations

$$A\mathbf{u}' = Q(\mathbf{u}) \tag{1.2}$$

for (1.1), including in particular the steady Boltzmann equation and cousins along with approximants such as BGK and discrete-velocity models [23,26]. Specifically, we are interested in the case when the linear operator A is self-adjoint, bounded, and one-to-one, but *not boundedly invertible*.

Following [23,26], we make the following assumptions on linear operator A and nonlinearity Q.

Hypothesis (H1)

(i) The linear operator A is bounded, self-adjoint, and one-to-one on the Hilbert space $\mathbb{H}$;