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Global existence and blow-up of solutions for the heat equation with exponential nonlinearity

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Abstract

We are concerned with the existence of global in time solution for a semilinear heat equation with exponential nonlinearity

$$\begin{cases} \partial_t u = \Delta u + e^u, & x \in \mathbf{R}^N, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{P}$$

where u_0 is a continuous initial function. In this paper, we consider the case where u_0 decays to $-\infty$ at space infinity, and study the optimal decay bound classifying the existence of global in time solutions and blowing up solutions for (P). In particular, we point out that the optimal decay bound for u_0 is related to the decay rate of forward self-similar solutions of $\partial_t u = \Delta u + e^u$.

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Keywords: Nonlinear heat equation; Exponential nonlinearity; Blow-up; Global in time existence; Forward self-similar solution

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1. Introduction

In this paper we consider a nonlinear heat equation

$$\begin{cases} \partial_t u = \Delta u + e^u, & x \in \mathbf{R}^N, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $\partial_t = \partial/\partial t$, $N \geq 1$ and u_0 is a continuous initial function. Throughout this paper, we assume that there exist constants $\epsilon \in (0,2)$ and $C > 0$ such that

$$-Ce^{|x|^{2-\epsilon}} \leq u_0(x) \leq C \quad \text{in } \mathbf{R}^N. \tag{1.2}$$

Let $T(u_0)$ be the maximal existence time of the unique classical solution of problem (1.1). Under the assumption (1.2), the supremum of the solution u of (1.1) on $\mathbf{R}^N$ is always bounded below as long as u exists, and if $T(u_0) < \infty$, the solution u satisfies

$$\limsup_{t \nearrow T(u_0)} \sup_{x \in \mathbf{R}^N} u(x,t) = +\infty.$$

Then we say that the solution u blows up in finite time and $T(u_0)$ is called the blow-up time of u. One of the most important questions for (1.1) is whether the global in time solution exists or not. Obviously, if u_0 is bounded below, then the solution u blows up in finite time by the comparison principle. Therefore the initial function u_0 has to diverge to $-\infty$ as $|x| \to \infty$. In this paper, we are concerned with the optimal decay rate of u_0 classifying global existence and blow-up of solutions for problem (1.1).

We first introduce some known results on global in time existence and blow-up of solutions for a semilinear heat equation with power type nonlinearity

$$\begin{cases} \partial_t u = \Delta u + u^p, & x \in \mathbf{R}^N, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \mathbf{R}^N, \end{cases} \tag{1.3}$$

where $p > 1$. For problem (1.3), it is well known that the exponent $p_F := 1 + 2/N$, which is called the Fujita exponent, plays an important role and (1.3) cannot possess nontrivial nonnegative global in time solutions if $1 < p \leq p_F$. In other words, if $u_0 \geq 0$ and $u_0 \not\equiv 0$, the solution must blow up in finite time. On the other hand, if $p > p_F$, there exist global in time solutions of (1.3) for suitable small initial data. See [1]. Furthermore, Lee and Ni in [7] proved the following:

- Assume that $p > p_F$ and $u_0(x)$ is of the form $\lambda\phi(x)$ for a parameter $\lambda > 0$, where $\phi \not\equiv 0$ is a nonnegative function satisfying

  $$\limsup_{|x| \to \infty} |x|^{\frac{2}{p-1}} \phi(x) < \infty.$$

  Then, if $\lambda > 0$ is sufficiently small, the solution u exists globally in time, that is, $T(u_0) = \infty$.