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On the rates of decay to equilibrium in degenerate and defective Fokker–Planck equations ☆

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Abstract

We establish sharp long time asymptotic behaviour for a family of entropies to defective Fokker–Planck equations and show that, much like defective finite dimensional ODEs, their decay rate is an exponential multiplied by a polynomial in time. The novelty of our study lies in the amalgamation of spectral theory and a quantitative non-symmetric hypercontractivity result, as opposed to the usual approach of the entropy method.

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Keywords: Fokker–Planck equations; Spectral theory; Non-symmetric hypercontractivity; Long time behaviour

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1. Introduction

1.1. Background

The study of Fokker–Planck equations (sometimes also called Kolmogorov forward equations) has a long history – going back to the early 20th century. Originally, Fokker and Planck used their equation to describe Brownian motion in a PDE form, rather than its usual SDE representation.

In its most general form, the Fokker–Planck equation reads as

$$\partial_t f(t,x) = \sum_{i,j=1}^{d} \partial_{x_i x_j} \big(D_{ij}(x) f(t,x)\big) - \sum_{i=1}^{d} \partial_{x_i} \left(A_i(x) f(t,x)\right), \tag{1.1}$$

with $t > 0, x \in \mathbb{R}^d$, and where $D_{ij}(x), A_i(x)$ are real valued functions, with $\mathbf{D}(x) = \big(D_{ij}(x)\big)_{i,j=1,\dots,d}$ being a positive semidefinite matrix.

The Fokker–Planck equation has many usages in modern mathematics and physics, with connection to statistical physics, plasma physics, stochastic analysis and mathematical finances. For more information about the equation, we refer the reader to [19]. Here we will consider a very particular form of (1.1) that allows degeneracies and defectiveness to appear.

1.2. The Fokker–Planck equation in our setting

In this work we will focus our attention on Fokker–Planck equations of the form:

$$\partial_t f(t,x) = Lf(t,x) := \operatorname{div}\left(\mathbf{D}\nabla f(t,x) + \mathbf{C}x f(t,x)\right), \qquad t > 0, x \in \mathbb{R}^d, \tag{1.2}$$

with appropriate initial conditions, where the matrix $\mathbf{D}$ (the *diffusion* matrix) and $\mathbf{C}$ (the *drift* matrix) are assumed to be constant and real valued.

In addition to the above, we will also assume the following:

(A) $\mathbf{D}$ is a positive semidefinite matrix with

$$1 \le r := \operatorname{rank}(\mathbf{D}) \le d.$$

(B) All the eigenvalues of $\mathbf{C}$ have positive real part (this is sometimes called *positively stable*).
(C) There exists no non-trivial $\mathbf{C}^T$-invariant subspace of $\operatorname{Ker}(\mathbf{D})$ (this is equivalent to *hypoellipticity* of (1.2), cf. [12]).

Each of these conditions has a significant impact on the equation:

- Condition (A) allows the possibility that our Fokker–Planck equation is degenerate ($r < d$).
- Condition (B) implies that the drift term confines the system. Hence it is crucial for the existence of a non-trivial steady state to the equation, and
- Condition (C) tells us that when $\mathbf{D}$ is degenerate, $\mathbf{C}$ compensates for the lack of diffusion in the appropriate direction and "pushes" the solution back to where diffusion happens.