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Overdetermined elliptic problems in topological disks ☆

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Abstract

We introduce a method, based on the Poincaré–Hopf index theorem, to classify solutions to overdetermined problems for fully nonlinear elliptic equations in domains diffeomorphic to a closed disk. Applications to some well-known nonlinear elliptic PDEs are provided. Our result can be seen as the analogue of Hopf's uniqueness theorem for constant mean curvature spheres, but for the general analytic context of overdetermined elliptic problems.

MSC: 35J25; 53A10

1. Introduction

The following famous theorem by Serrin [29] is widely regarded as the archetypal result on overdetermined problems for elliptic PDEs: *if* $u \in C^2(\overline{\Omega})$ *solves*

$$\begin{cases} \Delta u + 1 = 0 & \text{in} \quad \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial \nu} = c & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ *is a bounded smooth open domain, then* Ω *is a ball and* u *is radially symmetric*; here c is constant and ν is the interior unit normal of $\partial\Omega$. For the proof, Serrin introduced

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the *method of moving planes*, a version for overdetermined elliptic problems of the geometric Alexandrov theorem [1], according to which compact embedded constant mean curvature (CMC) hypersurfaces in $\mathbb{R}^{n+1}$ are round spheres.

Besides Alexandrov's theorem, the second classical result that models the geometry of compact CMC surfaces is Hopf's theorem [17,18]: *compact simply connected CMC surfaces in* $\mathbb{R}^3$ *are round spheres*. These two theorems are proved by totally different techniques, and complement each other. For instance, while Alexandrov's theorem works for arbitrary dimension and any topological type, Hopf's theorem is specific of dimension two (see Hsiang [19] for counterexamples in higher dimension) and needs the surface to be simply connected (Wente tori [34] are counterexamples for the multiply connected case). On the other hand, Hopf's theorem allows arbitrary self-intersections, and its proof provides important information on the local geometry of any CMC surface. Both results have been extremely influential in surface theory.

In this paper we prove what can be seen as a version for overdetermined elliptic problems of Hopf's theorem. Our theorem somehow completes the general parallelism between compact constant mean curvature theory and overdetermined elliptic problems in bounded domains initiated by Serrin in [29]. For unbounded domains Ω and semilinear elliptic PDEs, this parallelism has been deeply investigated, see e.g. [7,16,28,26,27,31]. Other connections of overdetermined elliptic problems and geometry can be found for instance in [8,10–12].

The following particular case can be seen as a model situation for our main result (Theorem 2.4). Consider the overdetermined problem

$$\begin{cases} F(D^2u, Du) = 0 & \text{in} \quad \Omega, \\ u = 0, \quad \dfrac{\partial u}{\partial \nu} = g(\nu) & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $F(D^2u, Du) = 0$ is a $C^{1,\alpha}$ fully nonlinear elliptic equation, $\Omega \subset \mathbb{R}^2$ is a bounded C^2 domain and $g \in C^1(\mathbb{S}^1)$. We will assume that the pair (F, g) satisfies the following compatibility condition, which ensures that (1.2) has solutions, and that we call *Property* $(*)$:

> *There is some solution* $u^0 \in C^2(\Omega_0^*)$ *to* $F(D^2u, Du) = 0$ *in some planar domain* Ω_0^*, *whose gradient is an orientation preserving diffeomorphism from* Ω_0^* *onto* $\mathbb{R}^2$, *such that* u^0 *solves* (1.2) *when restricted to some* C^2 *bounded domain* $\overline{\Omega_0} \subset \Omega_0^*$.

In this situation, and imposing a boundary regularity condition, we will prove: *if* Ω *is simply connected and* $u \in C^2(\overline{\Omega})$ *solves* (1.2)*, then up to a translation* $u = u^0$ *and* $\Omega = \Omega_0$.

There are many well-studied elliptic equations $F(D^2u, Du) = 0$ that satisfy Property $(*)$, and for which a solution to (1.2) was not previously known, not even for planar simply connected domains; examples will be provided in Section 4. An analogous theorem holds for elliptic equations of the more general form $F(D^2u, Du, u) = 0$, although in that case the corresponding Property $(*)$ is more involved; see Definition 2.1.

As happens with the classical situation of Alexandrov and Hopf, our theorem complements the previous methods for solving overdetermined elliptic problems (see e.g. [29,30,5,33]). Our method only works for the particular – but fundamental – case that Ω is a simply connected planar domain, since it depends on the Poincaré–Hopf index theorem. But on the other hand, it works for fully nonlinear elliptic equations without symmetries (the moving planes method is inapplicable in that situation), and for equations for which no *P-function* (in the sense of Weinberger's approach [33]) is known. Also, our method does not need to ensure, or impose,