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Existence and convexity of local solutions to degenerate Hessian equations

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Abstract

In this work, we prove the existence of convex solutions to the following k-Hessian equation

$$S_k[u] = K(y)g(y, u, Du)$$

in the neighborhood of a point $(y_0, u_0, p_0) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, where $g \in C^\infty$, $g(y_0, u_0, p_0) > 0$, $K \in C^\infty$ is nonnegative near y_0, $K(y_0) = 0$ and Rank $(D_y^2 K)(y_0) \geq n - k + 1$.

MSC: 35J60; 35J70

Keywords: Degenerate Hessian equations; Local solution; Convex solution; Nash–Moser–Hörmander iteration

1. Introduction

In this work, we study the following k-Hessian equation:

$$S_k[u] = f(y, u, Du), \tag{1.1}$$

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in an open domain $\Omega \subset \mathbb{R}^n$ with $2 \leq k \leq n$, where $f \geq 0$ is defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ with $f(y_0, u_0, p_0) = 0$. When $u \in C^2$, the k-Hessian operator $S_k[u]$ is defined by

$$S_k[u] = S_k(D^2u) = \sigma_k[\lambda(D^2u)] = \sum_{1 \leq i_1 < i_2 \ldots < i_k \leq n} \lambda_{i_1}\lambda_{i_2} \ldots \lambda_{i_k},$$

where $S_k(D^2u)$ is the sum of all the k-order principal minors of the Hessian matrix (D^2u), and $\lambda(D^2u) = (\lambda_1(D^2u), \ldots, \lambda_n(D^2u))$ are the eigenvalues of the matrix (D^2u). A motivation to study k-Hessian equations comes from the Christoffel–Minkowski problem, see [5–8] and references therein, also from calibrated geometries in [12]. The background of k-Hessian operator in terms of differential geometry can also be found in Section 4, [15].

When $f > 0$, the solutions u of (1.1) is considered with $\lambda(D^2u)$ in the so-called Gårding cone:

$$\Gamma_k(n) = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n;\ \sigma_j(\lambda) > 0, 1 \leq j \leq k\}.$$

If $f \geq 0$, the equation (1.1) is called degenerate, in this case, we consider the solutions with

$$\lambda(D^2u) \in \bar{\Gamma}_k(n) = \{\lambda \in \mathbb{R}^n;\ \sigma_j(\lambda) \geq 0, 1 \leq j \leq k\}.$$

A function $u \in C^2$ is called k-convex, if $\lambda(D^2u) \in \bar{\Gamma}_k(n)$. The n-convex functions are simply called convex.

The convexity of solutions of (1.1) is an important problem in the field of geometric analysis, including usual convexity, power convexity, log-convexity, or quasi-convexity. For example, in the study of the Christoffel–Minkowski problem (see [5–8]), the aim is to prove the existence of a convex body with prescribed area measure of suitable order, this is equivalent to prove the microscopic convexity principle (constant rank theorem) for some k-Hessian type equation on the unit sphere $\mathbb{S}^n$. There is also a strong connection between convexity properties of solutions to elliptic and parabolic partial differential equations and Brunn–Minkowski type inequalities for associated variational functionals, see [16–18]. When Guan [4] uses subsolution in place of all curvature restrictions on $\partial\Omega$ to construct local barriers for boundary estimates, it is an assumption that there exists a locally strictly convex function in $C^2(\overline{\Omega})$, see Theorem 1.1 and 1.3, [4].

The microscopic convexity principle, with applications in geometric equations on manifolds, has been established in [2] for very general fully nonlinear elliptic and parabolic operators of second order. Guan, Spruck and Xiao [9] point out that the asymptotic Plateau problem of finding a complete strictly locally convex hypersurface is reduced to a Dirichlet problem for a fully nonlinear equation, a special form of which is the k-Hessian equation, see Corollary 1.11, [9] and they have proved the existence of such hypersurfaces, it is especially interesting to notice that they have proved that, if $\partial\Omega$ is strictly (Euclidean) star-shaped about the origin, so is the unique solution, see Theorem 1.5, [9]. For the k-Hessian equation with $k = 2, n = 3$, the power convexity for Dirichlet problem of equation (1.1) with $f = 1$, and log-convexity for the eigenvalue problem have been studied in [16,17], see also [18]. The above convexity results are established on two facts, one is that the equations are elliptic, another is that the existence of classical (at least C^2) solution is already known or can be proved. However, in many important geometric problems, the associated k-Hessian equation is degenerate (see [11]), and for the degenerate elliptic k-Hessian equation, one can only prove the existence of $C^{1,1}$ solution for Dirichlet problem ([14]).