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ScienceDirect

J. Differential Equations 264 (2018) 6418–6458

Journal of Differential Equations

www.elsevier.com/locate/jde

Existence and instability of steady states for a triangular cross-diffusion system: A computer-assisted proof

Maxime Breden [a,b,c,*], Roberto Castelli [d]

[a] Technical University of Munich, Faculty of Mathematics, Research Unit "Multiscale and Stochastic Dynamics", 85748 Garching b. München, Germany

[b] CMLA, ENS Cachan, CNRS, Université Paris-Saclay, 61 avenue du Président Wilson, 94230 Cachan, France

[c] Département de Mathématiques et de Statistique, Université Laval, 1045 avenue de la Médecine, Québec, QC, G1V 0A6, Canada

[d] VU University Amsterdam, Department of Mathematics, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

Received 26 April 2017; revised 17 January 2018

Available online 1 February 2018

Abstract

In this paper, we present and apply a computer-assisted method to study steady states of a triangular cross-diffusion system. Our approach consist in an *a posteriori* validation procedure, that is based on using a fixed point argument around a numerically computed solution, in the spirit of the Newton–Kantorovich theorem. It allows to prove the existence of various non homogeneous steady states for different parameter values. In some situations, we obtain as many as 13 coexisting steady states. We also apply the *a posteriori* validation procedure to study the linear stability of the obtained steady states, proving that many of them are in fact unstable.

MSC: 35K59; 35Q92; 65G20; 65N35

Keywords: Rigorous numerics; Cross-diffusion; Steady states; Eigenvalue problem; Spectral analysis; Fixed point argument

* Corresponding author.

E-mail addresses: maxime.breden@tum.de (M. Breden), r.castelli3@gmail.com (R. Castelli).

https://doi.org/10.1016/j.jde.2018.01.033

1. Introduction

The primary goal of describing physical systems with mathematical models is to be able to explain and predict natural phenomena, within some range of approximation. In some circumstances the mathematical prediction and the experimental evidence don't agree, and a more trustful model is then required. Typically, one can add nonlinear or non homogeneous terms to get a more refined model, but this often seriously complicates the mathematical analysis of the system, which can become very hard, if not impossible, to study analytically. In this situation, numerical simulations allow insight of the phenomena and provide approximate, often very accurate, solutions. Aiming at formulating theorems, a powerful tool to validate approximate solutions into rigorous mathematical statements is provided by the rigorous computational techniques.

The diffusive Lotka–Volterra system, a well known model for population dynamics to study the competition between two species, is paradigmatic of the situation discussed above. It consists in the system

$$\begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + (r_1 - a_1 u - b_1 v)u, \quad \text{on } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + (r_2 - b_2 u - a_2 v)v, \quad \text{on } \mathbb{R}_+ \times \Omega, \\ \dfrac{\partial u}{\partial n} = 0 = \dfrac{\partial v}{\partial n}, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, and $u(t,x), v(t,x) \geq 0$ represent the population densities of two species at time t and position x. The non negative coefficients d_i, r_i, a_i and b_i $(i = 1, 2)$ describe the diffusion, the unhindered growth of the species, the intra-specific competition and the inter-specific competition respectively.

One of the fundamental problems is to determine if and under which assumptions the two species coexist, that converts into proving the existence or non-existence of stable positive equilibrium solutions. Several works have been produced to classify and analyze the stability of the equilibria for (1) and of related systems. We refer for instance to [38] for a short review. Of particular interest for our discussion is the result presented in [30]. If the domain Ω is convex, in that paper it is proved that any spatially non-constant equilibrium solution of (1) is unstable, if it exists. This implies that if the two species coexist, their densities must be homogeneous in the whole domain.

However, biological observations suggest that two competing species could coexist by forming pattern to avoid each other (a phenomenom called *spatial segregation*). Therefore we would like the model to exhibit stable non homogeneous steady states, but the quoted result shows that this is excluded (at least for convex domains). We point out that stable non homogeneous equilibria have been shown to exist for non convex domains [36], or for systems involving more than two species [29].

In the case of two competing species, to account for the expected stable inhomogeneous steady states, a generalization of (1) was proposed in [47]: