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# Finite-time blow-up for quasilinear degenerate Keller–Segel systems of parabolic–parabolic type

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**Abstract**

This paper deals with the quasilinear *degenerate* Keller–Segel systems of parabolic–parabolic type in a ball of $\mathbb{R}^N$ ($N \geq 2$). In the case of *non-degenerate* diffusion, Cieślak–Stinner [3,4] proved that if $q > m + \frac{2}{N}$, where $m$ denotes the intensity of diffusion and $q$ denotes the nonlinearity, then there exist initial data such that the corresponding solution blows up in *finite time*. As to the case of degenerate diffusion, it is known that a solution blows up if $q > m + \frac{2}{N}$ (see Ishida–Yokota [13]); however, whether the blow-up time is finite or infinite has been unknown. This paper gives an answer to the unsolved problem. Indeed, the *finite-time* blow-up of energy solutions is established when $q > m + \frac{2}{N}$.



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## 1. Introduction

In this paper we consider the following quasilinear degenerate Keller–Segel systems of parabolic–parabolic type:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m - \nabla \cdot (u^{q-1} \nabla v), & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u^m}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(\cdot, 0) = u_0,\ v(\cdot, 0) = v_0, & x \in \Omega \end{cases} \tag{KS}$$

in a ball

$$\Omega = B_R := \{x \in \mathbb{R}^N \mid |x| < R\},$$

where $N \geq 2, R > 0, m \geq 1, q \geq 2$, $\nu$ is the outer unit normal vector field to $\partial\Omega$ and the initial data are supposed to satisfy

$$u_0 \geq 0, \quad u_0 \in L^\infty(\Omega) \text{ with } \nabla u_0^m \in L^2(\Omega),$$
$$v_0 \geq 0, \quad v_0 \in W^{1,\infty}(\Omega).$$

Systems of this kind describe a part of the life cycle of cellular slime molds with the chemotaxis. More precisely, when cellular slime molds plunge into hunger, they move towards higher concentrations of the chemical substance secreted by cells. Moreover, these systems represent aggregation phenomenon not only in cellular slime molds but also in other creatures. Among them, when $m > 1$, the system (KS) is said to be *degenerate*. Then the system (KS) describes that a diffusion of cells depends only on own density and degenerates when $u = 0$. Usually $u(x, t)$ denotes the density of the cell and $v(x, t)$ represents the density of the semiochemical at point $x$ and time $t$. The minimal mathematical model (KS) with $m = 1$ and $q = 2$ has been introduced by Keller–Segel [14] in 1970. After that, the system has been simplified as parabolic–elliptic one with the second equation $0 = \Delta v - v + u$, and the model with nonlinear diffusion has been suggested in the survey [8] by Hillen–Painter; moreover, systems of this kind actively have been studied in the recent years.

From a mathematical point of view, it is a meaningful question whether solutions blow up or remain bounded. One of the reasons is that blow-up expresses the aggregation of cells and boundedness implies that the power of diffusion is stronger than that of chemotaxis. In the case of parabolic–elliptic type, there is a rich literature, e.g., Sugiyama–Kunii [23] and Sugiyama [22] for global existence and boundedness, [22] for finite-time blow-up, and Luckhaus–Sugiyama [16] and Ogawa [20] for decay property. In the case of minimal mathematical model, it is shown that the behavior of the corresponding solution is determined by the size of initial data. For details, when $N = 2$ and $\int_\Omega u_0 < 4\pi$ or when $\int_\Omega u_0 < 8\pi$ and $(u_0, v_0)$ are radially symmetric, Nagai–Senba–Yoshida [19] and Nagai–Ogawa [18] proved that the solution is global and bounded; when $N \geq 3$ and both $\|u_0\|_{L^{N/2+\delta}(\Omega)}$ and $\|v_0\|_{L^{N+\delta}(\Omega)}$ are sufficiently small for any $\delta \geq 0$, Winkler [25] (when $\delta > 0$) and Cao [2] (when $\delta = 0$) showed that the solution is global and bounded. On the other hand, if $N = 2$, Horstmann–Wang [9] observed that for almost all $M > 4\pi$ there are smooth

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