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Periodic solutions for one dimensional wave equation with bounded nonlinearity ☆

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Abstract

This paper is concerned with the periodic solutions for the one dimensional nonlinear wave equation with either constant or variable coefficients. The constant coefficient model corresponds to the classical wave equation, while the variable coefficient model arises from the forced vibrations of a nonhomogeneous string and the propagation of seismic waves in nonisotropic media. For finding the periodic solutions of variable coefficient wave equation, it is usually required that the coefficient $u(x)$ satisfies $\operatorname{ess\,inf} \eta_u(x) > 0$ with $\eta_u(x) = \frac{1}{2}\frac{u''}{u} - \frac{1}{4}\left(\frac{u'}{u}\right)^2$, which actually excludes the classical constant coefficient model. For the case $\eta_u(x) = 0$, it is indicated to remain an open problem by Barbu and Pavel (1997) [6]. In this work, for the periods having the form $T = \frac{2p-1}{q}$ (p, q are positive integers) and some types of boundary value conditions, we find some fundamental properties for the wave operator with either constant or variable coefficients. Based on these properties, we obtain the existence of periodic solutions when the nonlinearity is monotone and bounded. Such nonlinearity may cross multiple eigenvalues of the corresponding wave operator. In particular, we do not require the condition $\operatorname{ess\,inf} \eta_u(x) > 0$.

Keywords: Periodic solutions; Wave equation; Existence

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1. Introduction

The present work is devoted to the study of the existence of periodic solutions for the nonlinear wave equation

$$u(x)y_{tt} - (u(x)y_x)_x + u(x)g(y) = f(x,t), \quad x \in (0,1), \; t \in \mathbb{R}, \tag{1.1}$$

with the Sturm–Liouville boundary conditions

$$a_1 y(0,t) + b_1 y_x(0,t) = 0, \quad a_2 y(1,t) + b_2 y_x(1,t) = 0, \quad t \in \mathbb{R}, \tag{1.2}$$

and the periodic conditions

$$y(x,t+T) = y(x,t), \quad y_t(x,t+T) = y_t(x,t), \quad x \in (0,1), \; t \in \mathbb{R}, \tag{1.3}$$

where $u \in H^2(0,1)$ satisfies $u(x) \geq a > 0$ for all $x \in [0,1]$ and some positive constant a, g is a continuous, monotone (nondecreasing or nonincreasing) and bounded function, f is a given T-periodic function in t, and $a_i^2 + b_i^2 \neq 0$ for $i = 1, 2$.

It is obvious that (1.1) corresponds to the classical wave equation if the coefficient $u(x) \equiv C$ (a nonzero constant). The problem of finding periodic solutions to the constant coefficient wave equation has been extensively studied since the 1960s (see [1,2,7,8,10–14,16,18–21,26,35–37]). The first real breakthrough on this problem was due to Rabinowitz [35], where he rephrased the problem as a variational problem and obtained the existence of periodic solutions under the monotonicity assumption on the nonlinearity. Subsequently, many authors, such as Bahri, Brézis, Coron, Nirenberg etc., have used and developed Rabinowitz's variational methods to obtain related results, see [2,10–13]. In these works, the period T is required to be a rational multiple of the length of the spatial interval. The case in which T is some irrational multiple of the length of the spatial interval has also been investigated by Fečkan [22] and McKenna [33], where the frequencies are essentially the numbers whose continued fraction expansion is bounded (see [39]). At the end of the 1980s, a quite different approach which used the Kolmogorov–Arnold–Moser (KAM) theory was developed from the viewpoint of infinite dimensional dynamical systems by Kuksin [31] and Wayne [40]. This method allows one to obtain solutions whose periods are irrational multiples of the length of the spatial interval, and it also easily extends to construct quasi-periodic solutions, see [9,17,32,34,42]. However, unlike the variational techniques, it is local and restricted to equations with weak nonlinearity, or equivalently, to solutions of small amplitude. Later, in the original work of Craig and Wayne [20], the existence of periodic solutions for the one dimensional conservative nonlinear wave equation was also proved by using the Lyapunov–Schmidt method and Newton iterations.

On the other hand, if u is a function of x, then (1.1) is called the variable coefficient wave equation. Such a model arises from the forced vibrations of a nonhomogeneous string and the propagation of seismic waves in nonisotropic media (see [3–6,15,24,25,27–30,38]). More precisely, the vertical displacement $y(z,t)$ at depth z and time t of a plane seismic waves is described by the equation

$$\rho(z)y_{tt} - (\mu(z)y_z)_z = 0,$$

where ρ is the rock density and μ is the elasticity coefficient. By the change of variable