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Focal points and principal solutions of linear Hamiltonian systems revisited ☆

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Abstract

In this paper we present a novel view on the principal (and antiprincipal) solutions of linear Hamiltonian systems, as well as on the focal points of their conjoined bases. We present a new and unified theory of principal (and antiprincipal) solutions at a finite point and at infinity, and apply it to obtain new representation of the multiplicities of right and left proper focal points of conjoined bases. We show that these multiplicities can be characterized by the abnormality of the system in a neighborhood of the given point and by the rank of the associated T-matrix from the theory of principal (and antiprincipal) solutions. We also derive some additional important results concerning the representation of T-matrices and associated normalized conjoined bases. The results in this paper are new even for completely controllable linear Hamiltonian systems. We also discuss other potential applications of our main results, in particular in the singular Sturmian theory.

MSC: 34C10

Keywords: Linear Hamiltonian system; Proper focal point; Principal solution; Antiprincipal solution; Controllability

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1. Introduction

Oscillation properties of solutions is a widely studied subject in the theory of linear differential equations. In this paper we consider the linear Hamiltonian system

$$x' = A(t)\,x + B(t)\,u, \quad u' = C(t)\,x - A^T(t)\,u, \quad t \in \mathcal{I}, \tag{H}$$

where $\mathcal{I} \subseteq \mathbb{R}$ is a fixed interval and $A, B, C : \mathcal{I} \to \mathbb{R}^{n\times n}$ are given piecewise continuous matrix-valued functions on $\mathcal{I}$ such that $B(t)$ and $C(t)$ are symmetric and

$$B(t) \geq 0 \quad \text{for all } t \in \mathcal{I}, \tag{1.1}$$

i.e., the Legendre condition holds. Here $n \in \mathbb{N}$ is a given dimension. The purpose of the paper is twofold. We provide a new and unified theory of principal (and antiprincipal) solutions of (H) at a finite point $t_0 \in \mathcal{I}$ and at infinity, and use this generalized concept in order to characterize the multiplicities of focal points of conjoined bases of system (H). We will see that these two topics determine a new qualitative view on the principal solutions as a central object of the oscillation theory of differential systems.

As in our previous work [22–27] and [17,28,29], we do not impose any controllability (or normality) assumption on system (H). In [16, Theorem 3] and [8, Proof of Lemma 3.6(a)], Kratz and independently Fabbri, Johnson, and Núñez showed for this general context that for any conjoined basis (X, U) of (H) the kernel of $X(t)$ is piecewise constant on $\mathcal{I}$. Based on this fact Wahrheit defined in [30] a point $t_0 \in \mathcal{I} \setminus \{\inf \mathcal{I}\}$ to be a *left proper focal point* of (X, U) if $\operatorname{Ker} X(t_0^-) \subsetneqq \operatorname{Ker} X(t_0)$, with the multiplicity

$$m_L(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^-). \tag{1.2}$$

In a similar way we define $t_0 \in \mathcal{I} \setminus \{\sup \mathcal{I}\}$ to be a *right proper focal point* of (X, U) by the condition $\operatorname{Ker} X(t_0^+) \subsetneqq \operatorname{Ker} X(t_0)$, with the multiplicity

$$m_R(t_0) := \operatorname{def} X(t_0) - \operatorname{def} X(t_0^+). \tag{1.3}$$

The notations $\operatorname{Ker} X(t_0^\pm)$ and $\operatorname{def} X(t_0^\pm)$ represent the one-sided limits at t_0 of the piecewise constant quantities $\operatorname{Ker} X(t)$ and $\operatorname{def} X(t)$, being the kernel of $X(t)$ and its dimension. Note that in the (completely) controllable case the multiplicities of left and right proper focal points of (X, U) coincide with the defect of $X(t_0)$, as it is shown in [15, Theorem 4.1.3, pg. 126].

Let (X, U) be a conjoined basis (X, U) of (H). Under (1.1) we may choose an interval $(a, b) \subseteq \mathcal{I}$ with $a < b$ such that the matrix $X(t)$ has constant kernel on (a, b). Then we define the associated T-matrices

$$T_{\alpha,\,a^+} := \lim_{t\to a^+} S_\alpha^\dagger(t), \quad T_{\alpha,\,b^-} := \lim_{t\to b^-} S_\alpha^\dagger(t), \tag{1.4}$$

where $\alpha \in (a, b)$ is fixed and where the symmetric matrix $S_\alpha(t)$ is defined by

$$S_\alpha(t) := \int_\alpha^t X^\dagger(s)\,B(s)\,X^{\dagger\,T}(s)\,\mathrm{d}s, \quad t \in (a, b). \tag{1.5}$$