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The unsaturated flow in porous media with dynamic capillary pressure

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Abstract

In this paper we consider a degenerate pseudoparabolic equation for the wetting saturation of an unsaturated two-phase flow in porous media with dynamic capillary pressure-saturation relationship where the relaxation parameter depends on the saturation. Following the approach given in [13] the existence of a weak solution is proved using Galerkin approximation and regularization techniques. A priori estimates needed for passing to the limit when the regularization parameter goes to zero are obtained by using appropriate test-functions, motivated by the fact that considered PDE allows a natural generalization of the classical Kullback entropy. Finally, a special care was given in obtaining an estimate of the mixed-derivative term by combining the information from the capillary pressure with the obtained a priori estimates on the saturation.

MSC: 35K65; 35K70; 35Q35; 76S05

Keywords: Dynamic capillary pressure; Degenerate nonlinear parabolic PDE; Pseudoparabolic equations; Entropy; Existence of solutions

1. Introduction

Two-phase flow processes in porous media appear in many real-life situations, such as unsaturated water flow in the subsurface, remediation of contaminated sites and water–oil displacement during oil recovery. The physical–mathematical model of the two-phase flow consists of two

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mass balance equations, Darcy relations for the two phases and the capillary condition which is a constitutive relationship between the wetting-phase saturation and capillary pressure P_c. In this paper we include the dynamic capillary pressure relation proposed by Hassanizadeh and Grey [11] to the standard two-phase flow equations. This results in a transport equation containing higher-order mixed derivatives. Moreover, the obtained model represents a degenerate pseudoparabolic equation and our goal is to prove the global-in-time existence of its weak solutions. More specifically, we consider the equation

$$\partial_t S = \operatorname{div}\Big(a(S)\nabla\big(\partial_t \beta(S) - P_c(S)\big)\Big), \tag{1}$$

where

$$a(0) = a(1) = 0,$$
$$-P_c'(S) \to +\infty \text{ when } S \to 0, \text{ and } S \to 1,$$
$$\beta(S) = \int_0^S \tau(s)ds,$$
$$\tau(0) = 0,\ \tau(S) \text{ is bounded when } S \to 1 \text{ and } \tau \text{ is monotone increasing.}$$

The solution of equation (1) satisfies certain initial-boundary conditions which will be specified later.

We note that the nonequilibrium capillary effects in problems of enhancing oil and gas recovery from rocks were proposed in classical book by Barenblatt, Entov and Ryzhik [1], and then investigated by numerous scientists (see for example [2–4,16], [6,10]). The mathematical analysis of dynamic capillary pressure models for Richards' equation was intensively studied. A significant contribution has been done by Mikelić [13] where the existence, for any time interval, of an appropriate weak solution of the Richards' equation was proved. The existence of a weak solution to the Richards' equation with dynamic capillary pressure and hysteresis is studied by Schweizer [14]. For the two-phase flow model with dynamic capillary pressure and hysteresis, the existence of a weak solution is proved by Koch, Rätz and Schweizer [12] under assumption of non-degenerated mobilities. The first existence result for the two-phase flow model with dynamic capillary pressure and saturation dependent relaxation parameter is obtained by Cao and Pop in [7]. They considered the system in global pressure formulation with a strictly positive relaxation parameter and a constant Dirichlet boundary condition for the saturation.

In this work we consider the two-phase flow model with dynamic capillary pressure and degenerate saturation-dependent relaxation parameter. Moreover, we impose mixed boundary conditions with a non-constant boundary saturation. As our result shows (see also [13] and [7]), the existence theorem can be proved under certain relations between the orders of zeros of relative permeabilities and relaxation parameter and the order of singularities of capillary pressure function (see Proposition 3). Concerning these relations it is sufficient to analyze the case of the countercurrent imbibition flow instead of the full two-phase flow system. Therefore, we consider the countercurrent imbibition flow in the two-phase system that changes wettability (see [9]).

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