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Characterization for stability in planar conductivities ☆

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Abstract

We find a complete characterization for sets of uniformly strongly elliptic and isotropic conductivities with stable recovery in the L^2 norm when the data of the Calderón Inverse Conductivity Problem is obtained in the boundary of a disk and the conductivities are constant in a neighborhood of its boundary. To obtain this result, we present minimal a priori assumptions which turn out to be sufficient for sets of conductivities to have stable recovery in a bounded and rough domain. The condition is presented in terms of the integral moduli of continuity of the coefficients involved and their ellipticity bound as conjectured by Alessandrini in his 2007 paper, giving explicit quantitative control for every pair of conductivities.

MSC: 35R30; 35J15; 30C62

Keywords: Calderón Inverse Problem; Complex Geometric Optics Solutions; Stability; Quasiconformal mappings; Integral modulus of continuity

1. Introduction

Let γ be a strongly elliptic, isotropic conductivity coefficient in a bounded domain $\Omega \subset \mathbb{C}$, that is $\gamma : \mathbb{C} \to \mathbb{R}_+$ with $\operatorname{supp}(\gamma - 1) \subset \overline{\Omega}$ and both γ and its multiplicative inverse γ^{-1} bounded

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above by $K < \infty$ modulo null sets, which we summarize as

$$\gamma \in \mathcal{G}(K, \Omega).$$

For $1 \leq p \leq \infty$, the set $\mathcal{G}(K, \Omega)$ is a metric space when endowed with the L^p-distance

$$\mathrm{dist}_p^{\Omega}(\gamma_1, \gamma_2) = \|\gamma_1 - \gamma_2\|_{L^p(\Omega)}.$$

The conductivity inverse problem, proposed in 1980 by Alberto Calderón (see [14]), consists in determining γ from boundary measurements. This measurements are samples of the Dirichlet-to-Neumann (DtN) map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ which sends a function f to $\gamma \frac{\partial u}{\partial \nu}$, being ν the outward unit normal vector of $\partial\Omega$ and u the solution of the Dirichlet boundary value problem

$$\begin{cases} \nabla \cdot (\gamma \nabla u) = 0, \\ u_{|\partial\Omega} = f. \end{cases} \tag{1.1}$$

See Section 2.1 for the precise weak formulation of this equation and the DtN map.

In dimension 2, after the milestones [30] and [13], Astala and Päivärinta showed in [8] that every pair of conductivity coefficients $\gamma_1, \gamma_2 \in \mathcal{G}(K, \Omega)$ satisfies that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ if and only if $\gamma_1 = \gamma_2$. In higher dimensions, there are uniqueness results which require some a priori regularity of γ (see [19,23,26,12,36]).

Nevertheless, for the problem to be well-posed the inverse map $\Lambda_\gamma \mapsto \gamma$ should be continuous in some sense. Only then we have a chance that in real life situations, if our sample differs slightly from the DtN map of a certain body, then the solution we get is a good approximation of the conductivity of that body. Alessandrini showed in [4] that this is not possible in the L^∞-distance unless one imposes a priori conditions. In fact, G convergence gives explicit examples of highly oscillating sequences such that convergence of the DtN map does not imply the convergence of the conductivities in any dist_p^{Ω} distance (see [1,21]).

In [11] this problem was solved assuming that the space of conductivities is $L^\infty \cap C^\alpha(\Omega)$ by a cunning adaptation of Astala and Päivärinta arguments, to obtain stability in the L^∞-distance for Lipschitz domains. Later on, in [16] that result was extended to non-smooth conductivities, as long as they belong to a fractional Sobolev space $H^\alpha(\Omega)$ and showing stability with respect to the L^p-distance with $p < \infty$. Estimates for rough domains were obtained in [22]. Previous remarkable steps can be found in [5,28].

The purpose of the present paper is to characterize the subsets $\mathcal{F} \subset \mathcal{G}(K, \Omega)$ such that the inverse map is L^2-stable for these conductivities. The condition studied is to have a uniform integral modulus of continuity of exponent p for $1 \leq p < \infty$, under which L^2-stability is shown, the condition being necessary when $\Omega = \mathbb{D}$ and the conductivities are constant in a neighborhood of the boundary. Incidentally, every uniformly elliptic conductivity has a bounded integral modulus of continuity for every finite exponent.

Definition 1.1. An increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0} \omega(t) = 0$ is called *modulus of continuity*.

Let $f : \mathbb{R}^d \to \mathbb{C}$ be a measurable function and let $0 < p \leq \infty$. We define its integral modulus of continuity of exponent p, or p-modulus for short, as