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Dispersive estimates for massive Dirac operators in dimension two ☆

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Abstract

We study the massive two dimensional Dirac operator with an electric potential. In particular, we show that the t^{-1} decay rate holds in the $L^1 \to L^\infty$ setting if the threshold energies are regular. We also show these bounds hold in the presence of s-wave resonances at the threshold. We further show that, if the threshold energies are regular then a faster decay rate of $t^{-1}(\log t)^{-2}$ is attained for large t, at the cost of logarithmic spatial weights. The free Dirac equation does not satisfy this bound due to the s-wave resonances at the threshold energies.

1. Introduction

We consider the linear Dirac equation with potential,

$$i\partial_t \psi(x,t) = (D_m + V(x))\psi(x,t), \quad \psi(x,0) = \psi_0(x), \tag{1}$$

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with $\psi(x,t) \in \mathbb{C}^2$ when the spatial variable $(x_1, x_2) = x \in \mathbb{R}^2$. The free Dirac operator D_m is defined by

$$D_m = -i\alpha \cdot \nabla + m\beta = -i\alpha_1 \partial_{x_1} - i\alpha_2 \partial_{x_2} + m\beta.$$

Here $m \geq 0$ is the mass of the quantum particle. When $m > 0$, (1) is the massive Dirac equation and when $m = 0$, the equation is massless. The 2×2 Hermitian matrices α_j (with $\alpha_0 = \beta$) satisfy the anti-commutation relationship

$$\alpha_j \alpha_k + \alpha_k \alpha_2 = 2\delta_{jk} \mathbb{1}_{\mathbb{C}^2}, \qquad 0 \leq j, k \leq 2. \tag{2}$$

By convention, we take

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{3}$$

The hyperbolic system (1) was derived by Dirac to describe the evolution of a quantum particle at near luminal speeds. We view the system as a relativistic modification of the Schrödinger equation. This viewpoint is fruitful in light of the following identity,[1] which follows from (2):

$$(D_m - \lambda \mathbb{1})(D_m + \lambda \mathbb{1}) = (-i\alpha \cdot \nabla + m\beta - \lambda \mathbb{1})(-i\alpha \cdot \nabla + m\beta + \lambda \mathbb{1}) = (-\Delta + m^2 - \lambda^2). \tag{4}$$

This yields the identity

$$\mathcal{R}_0(\lambda) = (D_m + \lambda) R_0(\lambda^2 - m^2) \tag{5}$$

for the free Dirac resolvent, $\mathcal{R}_0(\lambda) = (D_m - \lambda)^{-1}$, where $R_0(\lambda) = (-\Delta - \lambda)^{-1}$ is the Schrödinger free resolvent and λ is in the resolvent set. We refer the reader to the text of Thaller, [33], for a more extensive introduction to the Dirac equation.

For the remainder of the paper $m > 0$. The massless Dirac equation, when $m = 0$, is of interest but requires a different approach to bound. Roughly speaking, solutions to the massless equation behave like solutions to the wave equation instead of the Klein–Gordon equation. There are significant differences in both the low energy and high energy behavior. In particular, the natural decay rate for the massless equation is $|t|^{-\frac{1}{2}}$, which cannot be improved by assuming additional regularity of the initial data as is done in the massive case. Analogous bounds for the perturbed Dirac equation in the massless case will be investigated elsewhere.

Our goal in this paper is to put the dispersive estimates for the massive ($m > 0$) Dirac equation on the same ground as those for the Schrödinger equation, [32,18,19]. To this end, we extend the recent results of the first two authors, [21], in two significant ways. First, we show that the dispersive bounds hold uniformly, that is we show that the $H^1 \to BMO$ bounds in [21] remain valid as operators from $L^1 \to L^\infty$. Second, we show a large time integrable bound holds at the cost of spatial weights. To state our results, we employ the following notation. Let $P_{ac}(H)$

[1] When we write scalar operators such as $-\Delta + m^2 - \lambda^2$, they are to be understood as $(-\Delta + m^2 - \lambda^2)\mathbb{1}_{\mathbb{C}^2}$. Similarly, we write L^p to indicate $L^p(\mathbb{R}^2) \times L^p(\mathbb{R}^2)$.