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Global solution branches for a nonlocal Allen–Cahn equation ☆

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Abstract

We consider the Neumann problem of a 1D stationary Allen–Cahn equation with nonlocal term. Our previous paper [4] obtained a local branch of asymmetric solutions which bifurcates from a point on the branch of odd-symmetric solutions. This paper derives the global behavior of the branch of asymmetric solutions, and moreover, determines the set of all solutions to the nonlocal Allen–Cahn equation. Our proof is based on a level set analysis for an integral map associated with the nonlocal term.

MSC: 34C23; 34B10; 37G40

Keywords: Allen–Cahn equation; Nonlocal term; Bifurcation; Monotonicity; Level set analysis

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1. Introduction

As a continuation of [4], this paper studies the following Neumann problem of a nonlinear ODE with nonlocal term:

$$\begin{cases} -du_{xx} = (1-u^2)\left(u - \dfrac{\mu}{2}\displaystyle\int_{-1}^{1} u\,\mathrm{d}x\right), & x \in I := (-1,1), \\ u_x(\pm 1) = 0, & \\ u_x(x) \geq 0, & x \in I, \end{cases} \tag{1.1}$$

where d is a positive parameter and μ is a nonnegative parameter. Such a class of Allen–Cahn type equations with nonlocal term has been studied in the field of the reaction–diffusion equations, (e.g., [3], [5], [6], [11]). Hence (1.1) has three constant solutions $u = 0$ and $u = \pm 1$. The reason why we focus on nondecreasing solutions is that any solution of (1.1) without the last condition can be constructed by connecting rescaled functions of monotone solutions. It is easy to verify that if $u(x)$ is a solution of (1.1), then $-u(-x)$ is also a solution.

Here we employ a fundamental but important reduction of (1.1) to the Neumann problem of a class of the Allen–Cahn equations

$$\begin{cases} -du_{xx} = (1-u^2)(u-\lambda), & x \in I, \\ u_x(\pm 1) = 0, & \\ u_x(x) \geq 0, & x \in I \end{cases} \tag{1.2}$$

with the nonlocal constraint

$$\lambda = \frac{\mu}{2}\int_{-1}^{1} u\,\mathrm{d}x. \tag{1.3}$$

Hence (1.1) is equivalent to the system of (1.2) and (1.3).

The purpose of this paper is to determine the set of nonconstant solutions of (1.2)–(1.3) (equivalently (1.1)):

$$\mathcal{S}(\mu) = \left\{ (u, \lambda, d) \in C^2_\nu(\overline{I}) \times \mathbb{R} \times \mathbb{R}_+ \,:\, u \text{ is a nonconstant solution of (1.2)–(1.3)} \right\} \tag{1.4}$$

for each $\mu > 0$, where $C^2_\nu(\overline{I}) := \{u \in C^2(\overline{I}) \,:\, u_x(\pm 1) = 0\}$. Hence any solution u with $\int_{-1}^{1} u\,\mathrm{d}x = 0$ of (1.2)–(1.3) satisfies a famous problem referred as the Chafee–Infante problem [2]:

$$\begin{cases} -du_{xx} = (1-u^2)u, & x \in I, \\ u_x(\pm 1) = 0, & \\ u_x(x) \geq 0, & x \in I. \end{cases} \tag{1.5}$$

Concerning (1.5), a detailed structure of the solution set

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