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Bifurcation of small limit cycles in cubic integrable systems using higher-order analysis ☆

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Abstract

In this paper, we present a method of higher-order analysis on bifurcation of small limit cycles around an elementary center of integrable systems under perturbations. This method is equivalent to higher-order Melinikov function approach used for studying bifurcation of limit cycles around a center but simpler. Attention is focused on planar cubic polynomial systems and particularly it is shown that the system studied by Żołądek (1995) [24] can indeed have eleven limit cycles under perturbations at least up to 7th order. Moreover, the pattern of numbers of limit cycles produced near the center is discussed up to 39th-order perturbations, and no more than eleven limit cycles are found.

MSC: 34C07; 34C23

Keywords: Bifurcation of limit cycles; Higher-order analysis; Darboux integral; Focus value

1. Introduction

Bifurcation theory of limit cycles is important for both theoretical development of qualitative analysis and applications in solving real problems. It is closely related to the well-known

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Hilbert's 16th problem [2], whose second part asks for the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that the following system,

$$\frac{dx}{dt} = P_n(x, y), \qquad \frac{dy}{dt} = Q_n(x, y), \tag{1}$$

can have, where $P_n(x, y)$ and $Q_n(x, y)$ represent n^{th}-degree polynomials in x and y. This problem has motivated many mathematicians and researchers in other disciplines to develop mathematical theories and methodologies in the areas of differential equations and dynamical systems. However, this problem has not been completely solved even for quadratic systems since Hilbert proposed the problem in the Second Congress of World Mathematicians in 1900. The maximal number of limit cycles obtained for some quadratic systems is 4 [3,4]. However, whether $H(2) = 4$ is still open. For cubic polynomial systems, many results have been obtained on the lower bound of the number of limit cycles. So far, the best result for cubic systems is $H(3) \geq 13$ [5,6]. Note that the 13 limit cycles obtained in [5,6] are distributed around several singular points.

When the problem is restricted to consider the maximum number of small-amplitude limit cycles, denoted by $M(n)$, bifurcating from a focus or a center in system (1), one of the best-known results is $M(2) = 3$, which was obtained by Bautin in 1952 [10]. For $n = 3$, a number of results in this research direction have been obtained. So far the best result for the number of small limit cycles around a focus is 9 [11–13], and that around a center is 12 [14].

One of powerful tools used for analyzing local bifurcation of limit cycles around a focus or a center is normal form theory (e.g., see [15–18]). Suppose system (1) has an elementary focus or an elementary center at the origin. With the computation methods using computer algebra systems (e.g., see [9,19–22]), we obtain the normal form expressed in polar coordinates as

$$\begin{aligned} \frac{dr}{dt} &= r\left(v_0 + v_1 r^2 + v_2 r^4 + \cdots + v_k r^{2k} + \cdots\right), \\ \frac{d\theta}{dt} &= \omega_c + \tau_0 + \tau_1 r^2 + \tau_2 r^4 + \cdots + \tau_k r^{2k} + \cdots, \end{aligned} \tag{2}$$

where r and θ represent the amplitude and phase of motion, respectively. v_k $(k = 0, 1, 2, \cdots)$ is called the kth-order focus value. v_0 and τ_0 are obtained from linear analysis. The first equation of (2) can be used for studying bifurcation and stability of limit cycles, while the second equation can be used to determine the frequency of the bifurcating periodic motion. Moreover, the coefficients τ_j can be used to determine the order or critical periods of a center (when $v_j = 0,\ j \geq 0$).

A particular attention has been paid to near-integrable polynomial systems, described in the form of [7,8]

$$\begin{aligned} \frac{dx}{dt} &= M^{-1}(x, y, \mu) H_y(x, y, \mu) + \varepsilon\, p(x, y, \varepsilon, \delta), \\ \frac{dy}{dt} &= -M^{-1}(x, y, \mu) H_x(x, y, \mu) + \varepsilon\, q(x, y, \varepsilon, \delta), \end{aligned} \tag{3}$$

where $0 < \varepsilon \ll 1$, μ and δ are vector parameters; $H(x, y, \mu)$ is an analytic function in x, y and μ; $p(x, y, \varepsilon, \delta)$ and $q(x, y, \varepsilon, \delta)$ are polynomials in x and y, and analytic in δ and ε. $M(x, y, \mu)$ is an integrating factor of the unperturbed system (3)$|_{\varepsilon=0}$.