ARTICLE IN PRESS YJDEQ:9146

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Classification of solutions of elliptic equations arising from a gravitational $O(3)$ gauge field model

Nari Choi, Jongmin Han *

Department of Mathematics, Kyung Hee University, Seoul 130-701, Republic of Korea

Received 16 July 2017

Abstract

In this paper, we study an elliptic equation arising from the self-dual Maxwell gauged $O(3)$ sigma model coupled with gravity. When the parameter τ equals 1 and there is only one singular source, we consider radially symmetric solutions. There appear three important constants: a positive parameter a representing a scaled gravitational constant, a nonnegative integer N_1 representing the total string number, and a nonnegative integer N_2 representing the total anti-string number. The values of the products $aN_1, aN_2 \in [0, \infty)$ play a crucial role in classifying radial solutions. By using the decay rates of solutions at infinity, we provide a complete classification of solutions for all possible values of aN_1 and aN_2. This improves previously known results.

MSC: 35J61; 35Q75; 81T13

Keywords: Topological solutions; Nontopological solutions; Gravitational Maxwell gauged $O(3)$ sigma model

1. Introduction

In this paper, we are interested in the following elliptic equation in $\mathbb{R}^2$:

$$\Delta v - \rho_\tau(x) f_\tau(v, a, \varepsilon) = 4\pi \sum_{j=1}^{d_1} n_{j,1}\delta_{p_{j,1}} - 4\pi \sum_{j=1}^{d_2} n_{j,2}\delta_{p_{j,2}}, \tag{1.1}$$

* Corresponding author.
E-mail addresses: nrchoi@khu.ac.kr (N. Choi), jmhan@khu.ac.kr (J. Han).

https://doi.org/10.1016/j.jde.2017.12.030
0022-0396/© 2017 Elsevier Inc. All rights reserved.

where

$$\rho_\tau(x) = \left[\prod_{j=1}^{d_1} |x - p_{j,1}|^{(1-\tau)n_{j,1}} \prod_{j=1}^{d_2} |x - p_{j,2}|^{(1+\tau)n_{j,2}}\right]^{-a},$$

$$f_\tau(v, a, \varepsilon) = \frac{2e^{\frac{(1-\tau)}{2}av}\Big[(1+\tau)e^v - (1-\tau)\Big]}{\varepsilon^2(1+e^v)^{1+a}}.$$

Here, $p_{j,1}$'s and $p_{j,2}$'s are disjoint points and called the strings and the antistrings, respectively. The unknown is

$$v : \mathbb{R}^2 \backslash \{p_{j,k} : j = 1, \cdots, d_k,\ k = 1, 2\} \to \mathbb{R}.$$

Furthermore, a is a nonnegative real number, $\tau \in [-1, 1]$ is a real number, δ_p denotes the Dirac measure concentrated at the point p, and $n_{j,k}$'s are positive integers representing the multiplicity of the strings and the antistrings $p_{j,k}$. We define the total string and anti-string numbers as

$$N_1 = n_{1,1} + \cdots + n_{d_1,1}, \quad N_2 = n_{1,2} + \cdots + n_{d_2,2}.$$

It is not difficult to check that v is a solution of (1.1) for τ if and only if $-v$ is a solution of (1.1) for $-\tau$ with the change of roles of $\{(p_{j,1}, n_{j,1}) : 1 \le j \le d_1\}$ and $\{(p_{j,2}, n_{j,2}) : 1 \le j \le d_2\}$. So, hereafter we assume that $0 \le \tau \le 1$.

The equation (1.1) arises from a self-dual gauge field model coupled with Einstein Equations. By taking into account a gravity in classical self-dual gauge field models, we need to solve the self-dual equations of models together with Einstein Equations. Recently, mathematical studies on these equations have grown up as interesting problems in various gauge filed theories [1–3,5, 9,14–16,19,20,23,25]. In particular, the equation (1.1) describes the Maxwell gauged $O(3)$ sigma model in the Bogomol'nyi regime on a space–time manifold. This model was introduced as an extension of Schroers' $U(1)$ Maxwell gauged harmonic map model [17,18] to a general relativity frame. The constant a stands for a scaled gravitational constant and the classical Schroers' model corresponds to the equation (1.1) with $a = 0$. For the detail background and derivation of (1.1), one may refer to [12,23–25].

From the physical motivation, it is natural to find solutions which gives the finite integration of the nonlinear term, that is $\rho_\tau(x) f_\tau(v, a, \varepsilon) \in L^1(\mathbb{R}^2)$. Then, the integrability condition yields three kinds of boundary conditions:

$$\begin{cases} \text{topological condition: } v(x) \ \to \ \sigma \in \mathbb{R} \quad \text{as} \quad |x| \to \infty, \\ \text{nontopological condition of type I: } v(x) \ \to \ -\infty \quad \text{as} \quad |x| \to \infty, \\ \text{nontopological condition of type II: } v(x) \ \to \ \infty \quad \text{as} \quad |x| \to \infty. \end{cases} \tag{1.2}$$

Solutions for each boundary condition are called topological solutions and nontopological solutions of type I and II, respectively. We often say a type I (resp. type II) solution simply for a nontopological solution of type I (resp. type II). The nature of solutions of (1.1) varies drastically according to the value $\tau \in [0, 1]$. In particular, we have different features of solutions according

Please cite this article in press as: N. Choi, J. Han, Classification of solutions of elliptic equations arising from a gravitational $O(3)$ gauge field model, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2017.12.030