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ScienceDirect
J. Differential Equations 264 (2018) 5109–5151
Journal of Differential Equations
www.elsevier.com/locate/jde

Stress regularity in quasi-static perfect plasticity with a pressure dependent yield criterion

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Received 6 February 2017; revised 6 September 2017
Available online 8 January 2018

Abstract

This work is devoted to establishing a regularity result for the stress tensor in quasi-static planar isotropic linearly elastic – perfectly plastic materials obeying a Drucker–Prager or Mohr–Coulomb yield criterion. Under suitable assumptions on the data, it is proved that the stress tensor has a spatial gradient that is locally squared integrable. As a corollary, the usual measure theoretical flow rule is expressed in a strong form using the quasi-continuous representative of the stress.

Keywords: Elasto-plasticity; Convex analysis; Quasi-static evolution; Regularity; Functions of bounded deformation; Capacity

1. Introduction

Perfect plasticity is a class of models in continuum solid mechanics involving a fixed threshold criterion on the Cauchy stress. When the stress is below a critical value, the underlying material behaves elastically, while the saturation of the constraint leads to permanent deformations after unloading back to a stress-free configuration. Elasto-plasticity represents a typical inelastic behavior, whose evolution is described by means of an internal variable, the plastic strain.

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https://doi.org/10.1016/j.jde.2017.12.034
0022-0396/

To formulate more precisely the problem, let us consider a bounded open set $\Omega \subset \mathbb{R}^n$ (in the following, only the dimension $n = 2$ will be considered), which stands for the reference configuration of an elasto-plastic body. In the framework of small strain elasto-plasticity the natural kinematic and static variables are the displacement field $u : \Omega \times [0, T] \to \mathbb{R}^n$ and the stress tensor $\sigma : \Omega \times [0, T] \to \mathbb{M}^{n\times n}_{\rm sym}$, where $\mathbb{M}^{n\times n}_{\rm sym}$ is the set of $n \times n$ symmetric matrices. In quasi-statics the equilibrium is described by the system of equations

$$-\mathrm{div}\,\sigma = f \quad \text{in } \Omega \times [0, T],$$

for some given body loads $f : \Omega \times [0, T] \to \mathbb{R}^n$. Perfect plasticity is characterized by the existence of a yield zone in which the stress is constrained to remain. The stress tensor must indeed belong to a given closed and convex subset K of $\mathbb{M}^{n\times n}_{\rm sym}$ with non empty interior:

$$\sigma \in K.$$

If σ lies inside the interior of K, the material behaves elastically, so that unloading will bring the body back to its initial configuration. On the other hand, if σ reaches the boundary of K (called the yield surface), a plastic flow may develop, so that, after unloading, a non-trivial permanent plastic strain will remain. The total linearized strain, denoted by $Eu := (Du + (Du)^T)/2$, is thus additively decomposed as

$$Eu = e + p.$$

The elastic strain $e : \Omega \times [0, T] \to \mathbb{M}^{n\times n}_{\rm sym}$ is related to the stress through the usual Hooke's law

$$\sigma := \mathbb{C}e,$$

where $\mathbb{C}$ is the symmetric fourth order elasticity tensor. The evolution of the plastic strain $p : \Omega \times [0, T] \to \mathbb{M}^{n\times n}_{\rm sym}$ is described by means of the flow rule

$$\dot p \in N_K(\sigma), \tag{1.1}$$

where $N_K(\sigma)$ is the normal cone to K at σ. From convex analysis, $N_K(\sigma) = \partial I_K(\sigma)$, *i.e.*, it coincides with the subdifferential of the indicator function I_K of the set K (where $I_K(\sigma) = 0$ if $\sigma \in K$, while $I_K(\sigma) = +\infty$ otherwise). Hence, from convex duality, the flow rule can be equivalently written as

$$\sigma : \dot p = \max_{\tau \in K} \tau : \dot p =: H(\dot p), \tag{1.2}$$

where $H : \mathbb{M}^{n\times n}_{\rm sym} \to [0, +\infty]$ is the support function of K. This last formulation (1.2) is nothing but Hill's principle of maximum plastic work, and $H(\dot p)$ denotes the plastic dissipation.

Standard models used for most of metals or alloys are those of Von Mises and Tresca. These kinds of materials are not sensitive to hydrostatic pressure, and plastic behavior is only generated through critical shearing stresses. In these models, if $\sigma_D := \sigma - \frac{\mathrm{tr}\sigma}{n}\mathrm{Id}$ stands for the deviatoric stress, the elasticity set K is of the form