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On a model for the Navier–Stokes equations using magnetization variables

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Abstract

It is known that in a classical setting, the Navier–Stokes equations can be reformulated in terms of so-called magnetization variables w that satisfy

$$\partial_t w + (\mathbb{P}w \cdot \nabla)w + (\nabla \mathbb{P}w)^\top w - \Delta w = 0, \tag{1}$$

and relate to the velocity u via a Leray projection $u = \mathbb{P}w$. We will prove the equivalence of these formulations in the setting of weak solutions that are also in $L^\infty(0, T; H^{1/2}) \cap L^2(0, T; H^{3/2})$ on the 3-dimensional torus.

Our main focus is the proof of global well-posedness in $H^{1/2}$ for a new variant of (1), where $\mathbb{P}w$ is replaced by w in the second nonlinear term:

$$\partial_t w + (\mathbb{P}w \cdot \nabla)w + \frac{1}{2}\nabla|w|^2 - \Delta w = 0. \tag{2}$$

This is based on a maximum principle, analogous to a similar property of the Burgers equations.

Keywords: Magnetization variables; Navier–Stokes equations; Burgers equations; Sobolev spaces; Maximum principle; Global well-posedness

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1. Introduction

The 3D Navier–Stokes equations model the flow of an incompressible viscous fluid and comprise the following system:

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = 0, \tag{1}$$

$$\nabla \cdot u = 0, \; u(0, x) = u_0(x). \tag{2}$$

Here the velocity $u(x, t)$ is an unknown evolving vectorfield and $p(x, t)$ is the unknown scalar pressure. The viscosity $\nu > 0$ will not play a significant role in our analysis, so we take $\nu = 1$ hereafter.

Global existence of weak solutions $u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; H^1)$ satisfying a certain energy inequality in $\mathbb{R}^3$ has been known since 1934, due to the fundamental contributions by Leray [14]. Since then there has been a great deal of progress, for example in the study of local well-posedness and global existence for small-data in certain critical spaces, as well as a number of important partial regularity results. However, the question of whether a function space exists in which we have global well-posedness for arbitrary initial data remains a major open problem.

For further discussion of some of the more well-known theory of the Navier–Stokes equations see, for example, [6], [9], [11], [13] and [21].

Given the challenge posed by the global well-posedness problem for this system, it can be useful to consider model problems. In this paper we present a natural model of the Navier–Stokes equations arising from the magnetization-variables formulation via a modification of one of the nonlinear terms that does not affect the scaling of the equations. For this system we can prove a global well-posedness result by virtue of a Burgers-type maximum principle.

The magnetization-variables formulation (also "Kuzmin–Oseledets" or "velicity" formulation) is more well known in the study of the Euler equations, but in the case of the Navier–Stokes system it has previously been discussed in, for example, [16] and [4]. Denoting the usual fluid velocity by u, this formulation comprises the following system, where w is called the magnetization variable:

$$\partial_t w + (u \cdot \nabla)w + (\nabla u)^\top w - \Delta w = 0 \tag{3}$$

$$u = \mathbb{P}w. \tag{4}$$

Here $\mathbb{P}$ denotes the Leray projection of L^2 onto L^2_σ, the closure of divergence-free functions. Unless stated otherwise, the analysis in this paper will take place under periodic boundary conditions, and the spatial domain will be denoted by

$$\mathbb{T}^3 := \mathbb{R}^3/2\pi\mathbb{Z}^3.$$

In Section 2 we will review the equivalence between the two formulations for classical solutions before proving new results about the correspondence in a weak setting. Specifically, in the context of weak solutions on $\mathbb{T}^3$ (which are defined below), we will show that for a weak solution $w \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ the projection $\mathbb{P}w$ is a weak solution of the Navier–Stokes equations. Constructing a solution w from a solution u of the Navier–Stokes equations is less straightforward, however we prove that if $u \in L^\infty(0, T; H^{1/2}) \cap L^2(0, T; H^{3/2})$ then there exists

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