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Hopf bifurcation in a delayed reaction–diffusion–advection population model ☆

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Abstract

In this paper, we investigate a reaction–diffusion–advection model with time delay effect. The stability/instability of the spatially nonhomogeneous positive steady state and the associated Hopf bifurcation are investigated when the given parameter of the model is near the principle eigenvalue of an elliptic operator. Our results imply that time delay can make the spatially nonhomogeneous positive steady state unstable for a reaction–diffusion–advection model, and the model can exhibit oscillatory pattern through Hopf bifurcation. The effect of advection on Hopf bifurcation values is also considered, and our results suggest that Hopf bifurcation is more likely to occur when the advection rate increases.

Keywords: Reaction–diffusion; Advection; Delay; Hopf bifurcation; Spatial heterogeneity

1. Introduction

During the past thirty years, delay induced instability has been investigated extensively for homogeneous reaction–diffusion equations with delay effect, and the spatial homogeneous and nonhomogeneous periodic solutions can occur through Hopf bifurcation. For models with the homogeneous Neumann boundary conditions, researchers were mainly concerned with the Hopf

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bifurcation near the constant positive equilibrium, see [10,14,16,19,20,23,27,29,34,35] and the references therein. For models with the homogeneous Dirichlet boundary conditions, the positive equilibrium is always spatially nonhomogeneous. Busenberg and Huang [3] first studied the Hopf bifurcation near such spatially nonhomogeneous positive equilibrium, and they found that, for the following prototypical single population model,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d\Delta u(x,t) + \lambda u(x,t)\left(1 - u(x,t-\tau)\right), & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.1}$$

time delay τ can make the unique spatially nonhomogeneous positive steady state unstable and induce Hopf bifurcation. Then, many authors investigated the Hopf bifurcation of models with the homogeneous Dirichlet boundary conditions, see [28,37,38,41,42]. Moreover, we refer to [9,11,21,22] and the references therein for the Hopf bifurcation of models with the nonlocal delay effect and the homogeneous Dirichlet boundary conditions.

In model (1.1), all the parameters are constant. However, due to the heterogeneity of the environment, the population may have a tendency to move up or down along the gradient of the habitats [2], which is referred to as taxis sometimes. Therefore, it might be more realistic to consider

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \nabla \cdot [d\nabla u - au\nabla m] + u(x,t)\left[m(x) - u(x,t-r)\right], & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \end{cases} \tag{1.2}$$

where $u(x,t)$ represents the population density at location x and time t, $d > 0$ is the diffusion coefficient, time delay $r > 0$ represents the maturation time, and Ω is a bounded domain in $\mathbb{R}^k$ $(1 \le k \le 3)$ with a smooth boundary $\partial\Omega$. Moreover, the intrinsic growth rate $m(x)$ is spatially dependent and may change sign, which means that, the intrinsic growth rate of the population is positive on favorable habitats and negative on unfavorable ones, and a measures the tendency of the population to move up or down along the gradient of $m(x)$. For $r = 0$, Cantrell and Cosner [4,5] investigated the effects of spatial heterogeneity on the dynamics of model (1.2) for the case of $a = 0$, and Belgacem and Cosner [2] considered the case of $a \ne 0$. For $\alpha = 0$, Shi et al. [36] showed that delay can induced Hopf bifurcation for model (1.2). We also refer to [6,12,13,26,30,31] and the references therein for the effects of spatial heterogeneity on single population and two competing populations models.

In this paper, we mainly investigate whether time delay r can induce Hopf bifurcation for reaction–diffusion–advection model (1.2). To our knowledge, there exist no concrete examples on Hopf bifurcation for reaction–diffusion–advection equations. As in [2], letting $v = e^{(-a/d)m(x)}u$, $t = \tilde{t}/d$, dropping the tilde sign, and denoting $\lambda = 1/d$, $\alpha = a/d$, $\tau = dr$, system (1.2) can be transformed as follows:

$$\begin{cases} \dfrac{\partial v}{\partial t} = e^{-\alpha m(x)}\nabla \cdot [e^{\alpha m(x)}\nabla v] + \lambda v\left[m(x) - e^{\alpha m(x)}v(x,t-\tau)\right], & x \in \Omega,\ t > 0, \\ v(x,t) = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.3}$$

Throughout the paper, unless otherwise specified, $m(x)$ satisfies the following assumption:

(A$_1$) $m(x) \in C^2(\overline{\Omega})$, and $\max_{x\in\overline{\Omega}} m(x) > 0$.

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