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Uniqueness of solutions for Keller–Segel system of porous medium type coupled to fluid equations

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Abstract

We prove the uniqueness of Hölder continuous weak solutions via duality argument and vanishing viscosity method for the Keller–Segel system of porous medium type equations coupled to the Stokes system in dimensions three. An important step is the estimate of the Green function of parabolic equations with lower order terms of variable coefficients, which seems to be of independent interest.

MSC: primary 35A02; secondary 35K65, 35Q92

Keywords: Keller–Segel equation; Porous media; Uniqueness; Duality argument; Vanishing viscosity; Green matrix of parabolic equation

1. Introduction

In this paper, we consider a mathematical model of the dynamics of swimming bacteria *Bacillus subtilis* in [17], where the movement of bacteria is formulated as a form of porous medium equation. More precisely, we are concerned with the following model

$$\partial_t \eta + v \cdot \nabla \eta - \Delta \eta^{1+\alpha} + \nabla \cdot \big(\chi(c)\eta^q \nabla c\big) = 0 \quad \text{in } \mathbb{R}^3_T, \tag{1.1a}$$

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$$\partial_t c + v\cdot\nabla c - \Delta c + \kappa(c)\eta = 0 \quad \text{in } \mathbb{R}^3_T, \tag{1.1b}$$

$$\partial_t v - \Delta v + \nabla p + \eta\nabla\phi = 0 \quad \text{in } \mathbb{R}^3_T, \tag{1.1c}$$

$$\nabla\cdot v = 0 \quad \text{in } \mathbb{R}^3_T, \tag{1.1d}$$

$$\eta(0,x) = \eta_0(x), \quad c(0,x) = c_0(x), \quad v(0,x) = v_0(x) \quad \text{in } \mathbb{R}^3, \tag{1.1e}$$

where $\alpha > 0$ and $q \geq 1$ are given constants, and $\mathbb{R}^3_T = (0,T)\times\mathbb{R}^3$. Here η, c, v, and p indicate biological cell density, the oxygen concentration, the fluid velocity, and the pressure, respectively. The nonnegative functions $k(c)$ and $\chi(c)$ denote the oxygen consumption rate and the chemotactic sensitivity, which are assumed to be locally bounded functions of c. Furthermore, the function ϕ is a time-independent potential function which indicates, for example, the gravitational force or centrifugal force. The system was proposed by Tuval et al. in [17] (see also [4]) for the case that $\alpha = 0$, $q = 1$ and fluid equations are the Navier–Stokes system (see e.g. [1–3,18,19] for related mathematical results).

Very recently, in [6, Theorem 1.8–Theorem 1.10], the existence of weak solutions and Hölder continuous weak solutions are proved under certain assumptions on $(\alpha, q, \chi, \kappa)$ (compare to [7, 10,16,20]). It is, however, unknown whether or not such solutions are unique. Our main motivation is to show the uniqueness of the Hölder continuous weak solutions of (1.1), for which we take the following simplified model by taking $\chi = 1$, $\kappa(c) = c$ and $q = 1$, since the system (1.1) is highly nonlinear and general χ, κ and $q > 1$ seem to be beyond our analysis (see Remark 2). So, we consider the following system of equations

$$\partial_t \eta + v\cdot\nabla\eta - \Delta\eta^{1+\alpha} + \nabla\cdot(\eta\nabla c) = 0 \quad \text{in } \mathbb{R}^3_T, \tag{1.2a}$$

$$\partial_t c + v\cdot\nabla c - \Delta c + c\eta = 0 \quad \text{in } \mathbb{R}^3_T, \tag{1.2b}$$

$$\partial_t v - \Delta v + \nabla p + \eta\nabla\phi = 0 \quad \text{in } \mathbb{R}^3_T, \tag{1.2c}$$

$$\nabla\cdot v = 0 \quad \text{in } \mathbb{R}^3_T, \tag{1.2d}$$

$$\eta(0,x) = \eta_0(x), \quad c(0,x) = c_0(x), \quad v(0,x) = v_0(x) \quad \text{in } \mathbb{R}^3. \tag{1.2e}$$

Since the system (1.2) satisfies the assumptions in [6, Theorem 1.10], it is straightforward that there exist Hölder continuous weak solutions when $\alpha > \frac{1}{8}$ and initial data are sufficiently regular. In this case, we can show that such solutions become unique. Our main result reads as follows:

Theorem 1.1. *Let $\alpha > \frac{1}{8}$ and (η_0, c_0, v_0) satisfy*

$$\eta_0(1 + |x| + |\ln\eta_0|) \in L^1(\mathbb{R}^3), \quad \eta_0 \in L^\infty(\mathbb{R}^3),$$
$$c_0 \in L^\infty(\mathbb{R}^3)\cap H^1(\mathbb{R}^3)\cap W^{1,m}(\mathbb{R}^3), \quad v_0 \in L^2(\mathbb{R}^3)\cap W^{1,m}(\mathbb{R}^3) \quad \textit{for any } m < \infty.$$

Then, Hölder continuous weak solutions of the system (1.2) are unique.

We note that there are some known results regarding uniqueness of Hölder continuous weak solutions of Keller–Segel equations of the porous medium type (see [12,14]).