ARTICLE IN PRESS YJDEQ:9155

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ●●● (●●●●) ●●●–●●●

Journal of Differential Equations

www.elsevier.com/locate/jde

On linear equations with general polynomial solutions

A. Laradji

Department of Mathematics & Statistics, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia

Received 8 September 2016; revised 22 September 2017

Abstract

We provide necessary and sufficient conditions for which an nth-order linear differential equation has a general polynomial solution. We also give necessary conditions that can directly be ascertained from the coefficient functions of the equation.

MSC: 34A05

1. Introduction

Consider the equation

$$p_0 y + p_1 y' + \cdots + p_{n-1} y^{(n-1)} + p_n y^{(n)} = 0 \tag{1}$$

where the p_k are functions (of a single variable x) continuous on some real interval in which p_n does not vanish. If this equation has n linearly independent polynomial solutions y_i $(1 \le i \le n)$, then an application of Cramer's rule to the system

$$\frac{p_0}{p_n} y_i + \cdots + \frac{p_{n-1}}{p_n} y_i^{(n-1)} = -y_i^{(n)} \quad (1 \le i \le n) \tag{2}$$

E-mail address: alaradji@kfupm.edu.sa.

https://doi.org/10.1016/j.jde.2018.01.002

shows that each $\frac{p_k}{p_n}$ is a rational function. We will therefore assume, without loss of generality, that the coefficients p_k in (1) are polynomials with no (non-constant) common factor and that p_n is monic.

Let K be the smallest integer k for which p_k in (1) is not the zero polynomial. Clearly, (1) has n linearly independent polynomial solutions if and only if the equation

$$p_K y + p_{K+1} y' + \cdots + p_{n-1} y^{(n-K-1)} + p_n y^{(n-K)} = 0$$

has $n-K$ linearly independent polynomial solutions. We can therefore assume that p_0 in (1) is not the zero polynomial. For notational convenience, we will write each p_h in (1) in "exponential" form: $p_h = \sum_{k\geq 0} \frac{p_{hk}}{k!} x^k$ with $p_h = 0$ if $h > n$. If $p_h \neq 0$, we denote its leading coefficient by γ_h (with $\gamma_n = 1$).

Determining conditions for which (1) has a fundamental set of polynomial solutions is a problem that has been discussed in several papers for various subclasses of (1). In [3], Calogero provided conditions for a wide class of second-order linear differential equations (with an arbitrary number of free parameters) to have general polynomial solutions. See also Calogero [4] in connection with a certain class of solvable N-body problems, Calogero [5] on the generalized hypergeometric equation, Calogero and Yi [6] concerning Jacobi polynomials and where para Jacobi polynomials are introduced, and Bagchi, Grandati and Quesne [2] where these polynomials are applied to the trigonometric Darboux–Pöschl–Teller potential.

The main objective in this note is to give conditions, which do not seem to be known, for which (1) and its nonhomogeneous counterpart have general polynomial solutions. In Proposition 2, we provide necessary conditions that can quickly be ascertained from the leading coefficients and degrees of the polynomials p_k. Propositions 4 and 5, while perhaps computationally more demanding, provide necessary and sufficient conditions. We end this note with a systematic way to construct nth-order linear equations containing an arbitrary number of parameters and having general polynomial solutions (cf. [3]).

2. Results

We will need the following lemma.

Lemma 1. *Let $r_1 < \cdots < r_n$ be a sequence of nonnegative integers and $y_1, \ldots, y_n$ be monic polynomials with respective degrees $d_1 < \cdots < d_n$. Consider the generalized Wronskian $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$, i.e. the determinant of the $n \times n$ matrix whose (i, j)th-element is $y_i^{(r_j)}$. Then, either $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$ is the zero polynomial or it has degree $\sum_{i=1}^n (d_i - r_i)$ and positive leading coefficient $\det\left(\begin{pmatrix} d_i \\ r_j \end{pmatrix}\right)_{1\leq i,j\leq n}$. Furthermore, if $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix} = 0$, then $W\begin{pmatrix} y_1, \ldots, y_n \\ s_1, \ldots, s_n \end{pmatrix} = 0$ for any sequence $s_1 < \cdots < s_n$ of nonnegative integers satisfying $r_i \leq s_i$ for all i.*

Proof. Clearly, the degree of $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$ does not exceed the sum $\sum_{i=1}^n (d_i - r_i)$ of the degrees of the diagonal polynomials in $W\begin{pmatrix} y_1, \ldots, y_n \\ r_1, \ldots, r_n \end{pmatrix}$, and the coefficient c of $x^{\sum_{i=1}^n (d_i - r_i)}$ is

Please cite this article in press as: A. Laradji, On linear equations with general polynomial solutions, J. Differential Equations (2018), https://doi.org/10.1016/j.jde.2018.01.002