ARTICLE IN PRESS YJDEQ:9125

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Journal of Differential Equations

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Energy decay rates for solutions of the wave equations with nonlinear damping in exterior domain

M. Daoulatli [a,b,*]

[a] *Imam Abdulrahman Bin Faisal University, Saudi Arabia*
[b] *University of Carthage, Tunisia*

Received 8 July 2017; revised 29 November 2017

Abstract

In this paper we study the behaviors of the energy of solutions of the wave equations with localized nonlinear damping in exterior domains.

MSC: primary 35L05, 35B40; secondary 35L70, 35B35

Keywords: Wave equation; Nonlinear damping; Decay rate; Exterior domain

1. Introduction and statement of the results

Let O be a compact domain of $\mathbb{R}^d$ $(d \geq 1)$ with C^∞ boundary Γ and $\Omega = \mathbb{R}^d \backslash O$. Consider the following wave equation with localized nonlinear damping

$$\begin{cases} \partial_t^2 u - \Delta u + a(x) \left|\partial_t u\right|^{r-1} \partial_t u = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}_+ \times \Gamma, \\ u(0,x) = u_0 \quad \text{and} \quad \partial_t u(0,x) = u_1, & \end{cases} \tag{1.1}$$

* Correspondence to: Imam Abdulrahman Bin Faisal University, Saudi Arabia.
E-mail address: mdaoulatli@iau.edu.sa.

https://doi.org/10.1016/j.jde.2017.12.009

here Δ denotes the Laplace operator in the space variables. $a(x)$ is a nonnegative function in $L^{\infty}(\Omega)$. Throughout this paper we assume that $1 < r \leq 1 + \frac{2}{d}$. Below $r_0 > 0$ is a fixed constant such that $O \subset B_{r_o} = \{x \in \mathbb{R}^d; |x| < r_0\}$.

The existence and uniqueness of global solutions to the problem (1.1) is standard (see [16]). If (u_0, u_1) is in $H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$, then the system (1.1), admits a unique solution u in the class

$$u \in C^0\Big(\mathbb{R}_+, H_0^1(\Omega)\Big) \cap C^1\Big(\mathbb{R}_+, L^2(\Omega)\Big) \text{ and } \partial_t u \in L^{\infty}\Big(\mathbb{R}_+, H_0^1(\Omega)\Big) \cap W^{1,\infty}\Big(\mathbb{R}_+, L^2(\Omega)\Big).$$

Let us consider the energy at instant t defined by

$$E_u(t) = \frac{1}{2}\int_{\Omega} \left(|\nabla u(t,x)|^2 + |\partial_t u(t,x)|^2\right) dx.$$

The energy functional satisfies the following identity

$$E_u(T) + \int_0^T \int_{\Omega} a(x)|\partial_t u|^{r+1} dxdt = E_u(0), \tag{1.2}$$

for every $T \geq 0$. Moreover, we have

$$\begin{aligned} &\|\nabla \partial_t u\|^2_{L^{\infty}(\mathbb{R}_+, L^2(\Omega))} + \left\|\partial_t^2 u\right\|^2_{L^{\infty}(\mathbb{R}_+, L^2(\Omega))} \\ &\leq 2(1 + \|a\|_{L^{\infty}})\left(\|u_0\|^2_{H^2} + \|u_1\|^2_{H^1} + \|u_1\|^{2r}_{H^1}\right). \end{aligned} \tag{1.3}$$

The study of the behaviors of the energy decay of solutions of the damped wave equation has a very long history. First we give a summary of results on the asymptotic behavior of the energy of solutions of the nonlinear system (1.1) in the free space $\mathbb{R}^d$ and for a globally distributed damping. For the Klein–Gordon equation with localized nonlinear damping, under the Lion's condition a polynomial decay rate is derived by Nakao [19] for compactly supported initial data and he show in this case that

$$E_u(t) \leq C(1+t)^{-\gamma}, \text{ if } 1 < r < 1 + \frac{2}{d}, \tag{1.4}$$

where $\gamma = \frac{2+d-dr}{r-1}$ and

$$E_u(t) \leq C(\ln(2+t))^{-d}, \text{ if } r = 1 + \frac{2}{d}. \tag{1.5}$$

Mochizuki and Motai [17] give a decay rate estimate for weighted initial data. More precisely, they show that if $1 < r < 1 + \frac{2}{d}$, the energy decays according to

$$E_u(t) \leq C(1+t)^{-\gamma}, \text{ where } 0 < \gamma < \frac{2+d-dr}{r-1} \text{ and } \gamma \leq 1. \tag{1.6}$$