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Stationary convection–diffusion equation in an infinite cylinder

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Abstract

We study the existence and uniqueness of a solution to a linear stationary convection–diffusion equation stated in an infinite cylinder, Neumann boundary condition being imposed on the boundary. We assume that the cylinder is a junction of two semi-infinite cylinders with two different periodic regimes. Depending on the direction of the effective convection in the two semi-infinite cylinders, we either get a unique solution, or one-parameter family of solutions, or even non-existence in the general case. In the latter case we provide necessary and sufficient conditions for the existence of a solution.

MSC: primary 35B27; secondary 76M50, 78M40

Keywords: Convection–diffusion equation; Infinite cylinder; Stabilization at infinity; Effective drift

0. Introduction

The paper deals with a stationary linear convection–diffusion equation in an infinite cylinder $\mathbb{G} = (-\infty, \infty) \times Q$ with a Lipschitz bounded domain $Q \subset \mathbb{R}^{d-1}$, at the cylinder boundary the Neumann condition being imposed. We assume that, except for a compact set in $\mathbb{G}$, the coefficients of the convection–diffusion operator are periodic in x_1 both in the left and in the right half cylinder. These two periodic operators need not coincide. This problem reads

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$$\begin{cases} -\mathrm{div}\,(a(x)\,\nabla\,u(x)) + b(x)\cdot\nabla u(x) = f(x), & x\in\mathbb{G},\\ a(x)\nabla u(x)\cdot n = g(x), & x\in\Sigma. \end{cases} \tag{1}$$

Under uniform ellipticity assumptions we study if this problem has a bounded solution and if such a solution is unique. Concerning the functions f and g we assume that they decay fast enough as $|x_1|\to\infty$. Following [5] one can introduce the so-called effective axial drifts $\bar b^+$ and $\bar b^-$ in the right and left halves of the cylinder, respectively. It turns out that the mentioned existence and uniqueness issues depend on the signs of $\bar b^+$ and $\bar b^-$ (both effective drifts can be positive, or negative, or zero).

The main result of the paper is summarized below.

If $\bar b^+<0$ and $\bar b^->0$, then for any two constants K^- and K^+ there is a solution of (1) that converges to K^- as $x_1\to-\infty$ and to K^+ as $x_1\to+\infty$.

If $\bar b^+\geq 0$ and $\bar b^->0$ or $\bar b^+<0$ and $\bar b^-\leq 0$ then a bounded solution exists and is unique up to an additive constant.

The case $\bar b^+\geq 0$ and $\bar b^-\leq 0$ is more interesting. In this case a bounded solution need not exist. We will show that in this case the problem adjoint to (1) has a bounded solution $p\in C(\bar{\mathbb{G}})$, which is positive under proper normalization. Then problem (1) has a bounded solution if and only if

$$\int\limits_{\mathbb{G}} f(x)p(x)\,dx + \int\limits_{\Sigma} g(x)p(x)d\sigma = 0. \tag{2}$$

A bounded solution in this case is unique up to an additive constant.

The qualitative behavior of the function p in the two semi-infinite cylinders varies depending on whether the effective drift in that cylinder is equal to zero or not. Namely, if $\bar b^+<0$ and $\bar b^->0$, then p decays exponentially as $x_1\to\infty$. If, however, the effective drift is zero in one of the semi-infinite cylinders, p will stabilize to a periodic regime in that part, as $x_1\to\infty$.

In all three cases any bounded solution converges to some constants as $|x_1|\to\infty$. Moreover, this convergence has exponential rate if $f(x)$ and $g(x)$ decay exponentially as $|x_1|\to\infty$.

The question of the behavior at infinity of solutions to elliptic equations in cylindrical and conical domains attracted the attention of mathematicians since the middle of 20th century. In [8] it was shown that for a divergence form elliptic operator in a semi-infinite cylinder there is a unique (up to an additive constant) bounded solution. It stabilizes to a constant at infinity. Similar problem for a convection–diffusion operator has been studied in [9], [5]. In these works necessary and sufficient conditions for the uniqueness of a bounded solution were provided. The work [7] deals with a particular class of convection-diffusion equations in an infinite cylinder. In [10], [11] and [12] specific classes of semi-linear elliptic equations in a half-cylinder were considered. It was shown in particular that a global solution, if it exists, decays at least exponentially with large axial distance. The behavior at infinity of solutions to some classes of elliptic systems, in particular to linear elasticity was investigated in [13]. In [14] the uniqueness issue was studied for solutions of second order elliptic equations in unbounded domains under some dissipation type assumptions on the coefficients. The work [15] deals with solutions of elliptic systems in a cylinder that have a bounded weighted Dirichlet integral. Paper [16] studies the existence of solutions of symmetric elliptic systems in weighted spaces with exponentially growing or decaying weights.

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