ARTICLE IN PRESS YJDEQ:9139

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J. Differential Equations ●●● (●●●●) ●●●–●●●

Journal of Differential Equations

www.elsevier.com/locate/jde

p-Euler equations and p-Navier–Stokes equations

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Received 18 May 2017; revised 24 October 2017

Abstract

We propose in this work new systems of equations which we call p-Euler equations and p-Navier–Stokes equations. p-Euler equations are derived as the Euler–Lagrange equations for the action represented by the Benamou–Brenier characterization of Wasserstein-p distances, with incompressibility constraint. p-Euler equations have similar structures with the usual Euler equations but the 'momentum' is the signed $(p-1)$-th power of the velocity. In the 2D case, the p-Euler equations have streamfunction-vorticity formulation, where the vorticity is given by the p-Laplacian of the streamfunction. By adding diffusion presented by γ-Laplacian of the velocity, we obtain what we call p-Navier–Stokes equations. If $\gamma = p$, the *a priori* energy estimates for the velocity and momentum have dual symmetries. Using these energy estimates and a time-shift estimate, we show the global existence of weak solutions for the p-Navier–Stokes equations in $\mathbb{R}^d$ for $\gamma = p$ and $p \geq d \geq 2$ through a compactness criterion.

Keywords: Wasserstein-p geodesics; Benamou–Brenier functional; p-momentum; p-Laplacian; Global weak solutions

1. Introduction

The Wasserstein distances [1–4] for probability measures in a domain $O \subset \mathbb{R}^d$ are closely related to optimal transport and are useful for image processing [5], machine learning [6] and fluid mechanics [7]. If O is convex and bounded, the Wasserstein-p ($p > 1$) distance between

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https://doi.org/10.1016/j.jde.2017.12.023

Please cite this article in press as: L. Li, J.-G. Liu, p-Euler equations and p-Navier–Stokes equations, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.12.023

two probability measures μ, ν in O can be reformulated as the following optimization problem [4, Sec. 5.4]:

$$W_p^p(\mu,\nu) = \min_{\rho,m}\left\{\int_0^1 p\mathscr{B}_p(\rho,m)dt : \partial_t\rho + \nabla\cdot m = 0, \rho|_{t=0} = \mu, \rho|_{t=1} = \nu\right\},$$

where ρ is a nonnegative measure and m is a vector measure, both of which are time-dependent. $\mathscr{B}_p$ is the Benamou–Brenier functional, and see Equation (2) for the expression in the case $m \ll \rho$ (i.e. m is absolutely continuous with respect to ρ). This Benamou–Brenier characterization of Wasserstein distances provides a least action principle framework for us to study Wasserstein geodesics.

In applications like image processing, one usually wants to find the geodesics between two shapes using some suitable action [5]. In [8], Liu et al. considered two shapes (open connected sets) Ω_0 and Ω_1 in O with equal volume $|\Omega_0| = |\Omega_1|$. Assigning the two shapes with uniform probability measures

$$\mu = \frac{1}{|\Omega_0|}\chi(\Omega_0),\ \ \nu = \frac{1}{|\Omega_1|}\chi(\Omega_1),$$

where $\chi(E)$ for a set E means the characteristic function, the Wasserstein-p distance between these two shapes is defined as the Wasserstein-p distance between μ and ν. The authors of [8] considered the geodesics between Ω_0 and Ω_1 with the action represented by the Benamou–Brenier characterization of Wasserstein-2 distance under incompressibility constraint. In other words, they studied the action

$$\mathcal{A} = \frac{1}{2}\int_0^1\int_O \rho|v|^2\,dxdt,$$

with the constraint

$$|\Omega_t| = |\Omega_0|,\ \ \rho(\cdot,t) = \frac{1}{|\Omega_t|}\chi(\Omega_t),\ \ \forall t\in[0,1].$$

Note that the action $\mathcal{A}$ here is different from the one used in [8] by a multiplicative constant, to be consistent with our convention in this paper. They found that the Euler–Lagrange equations for the geodesics under this constraint are the incompressible, irrotational Euler equations with free boundary. They proved that the distance between two shapes under this notion with incompressibility constraint is equal to the Wasserstein-2 distance.

The work in [8] is related to Arnold's least action principle [9], where Arnold discovered that the Euler equations of inviscid fluid flow can be viewed as the geodesic path in the group of volume-preserving diffeomorphisms in a fixed domain. One of the differences is that the equations in [8] are irrotational with free boundary. The free boundary problems for incompressible Euler equations are waterwave equations which have attracted a lot of attention [10–14]. In [12, 13], Wu proved the wellposedness of waterwave problems in Sobolev spaces with general data for irrotational, no surface tension cases. In [14], Shatah and Zeng solved the zero tension limit

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