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Continuity of pullback and uniform attractors

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Abstract

We study the continuity of pullback and uniform attractors for non-autonomous dynamical systems with respect to perturbations of a parameter. Consider a family of dynamical systems parameterized by $\lambda \in \Lambda$, where Λ is a complete metric space, such that for each $\lambda \in \Lambda$ there exists a unique pullback attractor $\mathcal{A}_\lambda(t)$. Using the theory of Baire category we show under natural conditions that there exists a residual set $\Lambda_* \subseteq \Lambda$ such that for every $t \in \mathbb{R}$ the function $\lambda \mapsto \mathcal{A}_\lambda(t)$ is continuous at each $\lambda \in \Lambda_*$ with respect to the Hausdorff metric. Similarly, given a family of uniform attractors $\mathbb{A}_\lambda$, there is a residual set at which the map $\lambda \mapsto \mathbb{A}_\lambda$ is continuous. We also introduce notions of equi-attraction suitable for pullback and uniform attractors and then show when Λ is compact that the continuity of pullback attractors and uniform attractors with respect to λ is equivalent to pullback equi-attraction and, respectively, uniform equi-attraction. These abstract results are then illustrated in the context of the Lorenz equations and the two-dimensional Navier–Stokes equations.

Keywords: Pullback attractor; Uniform attractor

1. Introduction

The theory of attractors plays an important role in understanding the long time behavior of dynamical systems, see Babin and Vishik [1], Billotti and LaSalle [3], Chueshov [7], Hale [11],

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Ladyzhenskaya [17], Robinson [22] and Temam [27]. For the autonomous theory, we consider a family of dissipative dynamical systems parameterized by Λ such that for each $\lambda \in \Lambda$ the corresponding dynamical system possesses a unique compact global attractor $\mathcal{A}_\lambda \subseteq Y$, where Y is a complete metric space with metric d_Y. Under very mild assumptions (see for example [12] and the references therein) the map $\lambda \mapsto \mathcal{A}_\lambda$ is known to be upper semicontinuous. This means that

$$\rho_Y(\mathcal{A}_\lambda, \mathcal{A}_{\lambda_0}) \to 0 \qquad \text{as} \qquad \lambda \to \lambda_0$$

where $\rho_Y(A, C)$ denotes the Hausdorff semi-distance

$$\rho_Y(A, C) = \sup_{a \in A} \inf_{c \in C} d_Y(a, c). \tag{1.1}$$

However, lower semicontinuity

$$\rho_Y(\mathcal{A}_{\lambda_0}, \mathcal{A}_\lambda) \to 0 \qquad \text{as} \qquad \lambda \to \lambda_0,$$

and hence full continuity with respect to the Hausdorff metric, is much harder to prove.

For autonomous systems, general results on lower semicontinuity require strict conditions on the structure of the unperturbed global attractor, which are rarely satisfied for complicated systems (see Hale and Raugel [13] and Stuart and Humphries [25]). However, Babin and Pilyugin [2] and Hoang et al. [14] showed, using the theory of Baire category, that continuity holds for λ_0 in a residual set $\Lambda_* \subseteq \Lambda$ under natural conditions when Λ is a complete metric space. We recall this result for autonomous systems as Theorem 1.1 below.

Let Λ and X be complete metric spaces. We will suppose that $S_\lambda(\cdot)$ is a parameterized family of semigroups on X for $\lambda \in \Lambda$ that satisfies the following properties:

(G1) $S_\lambda(\cdot)$ has a global attractor $\mathcal{A}_\lambda$ for every $\lambda \in \Lambda$;
(G2) there is a bounded subset D of X such that $\mathcal{A}_\lambda \subseteq D$ for every $\lambda \in \Lambda$; and
(G3) for $t > 0$, $S_\lambda(t)x$ is continuous in λ, uniformly for x in bounded subsets of X.

Note that condition (G2) can be strengthened and (G3) weakened by replacing *bounded* by *compact*. These modified conditions will be referred to as conditions (G2′) and (G3′).

Theorem 1.1. *Under assumptions* (G1–G3) *above—or under the assumptions* (G1), (G2′) *and* (G3′)*—$\mathcal{A}_\lambda$ is continuous in λ at all λ_0 in a residual subset of Λ. In particular the set of continuity points of $\mathcal{A}_\lambda$ is dense in Λ.*

The proof developed in [14] of the above theorem, which appears there as Theorem 5.1, is more direct than previous proofs (e.g. in [2]) and can be modified to establish analogous results for the pullback attractors and uniform attractors of non-autonomous systems. This is the main purpose of the present paper. After briefly introducing some definitions and notations concerning attractors and Baire category theory in Section 2, in Section 3 we prove Theorem 3.3, our main result concerning pullback attractors. Section 4 then contains Theorem 4.1, which provides similar results for uniform attractors. In addition, we investigate the continuity of pullback and uniform attractors on the entire parameter space Λ. It was proved by Li and Kloeden [19] (see

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