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Upper bound on the slope of steady water waves with small adverse vorticity ☆

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Abstract

We consider the angle of inclination (with respect to the horizontal) of the profile of a steady 2D inviscid symmetric periodic or solitary water wave subject to gravity. There is an upper bound of 31.15° in the irrotational case [1] and an upper bound of 45° in the case of favorable vorticity [13]. On the other hand, if the vorticity is adverse, the profile can become vertical. We prove here that if the adverse vorticity is sufficiently small, then the angle still has an upper bound which is slightly larger than 45°.

1. Introduction

The extreme Stokes wave has angle θ (with respect to the horizontal) exactly 30° at its crest, as originally conjectured by Stokes himself [12]. This is the limiting wave, singular at its crest, of the curve $\mathscr{K}$ of irrotational waves that bifurcates from a trivial (flat laminar) wave. However, it was a surprise when numerical calculations [11,8,3] indicated that the angle θ surpasses 30° for some waves on $\mathscr{K}$ that are very close to the extreme wave. This fact was subsequently proven by McLeod in [9]. The maximum angle does not occur at the crest but very close to it.

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In a remarkable paper Amick [1] proved that for any irrotational wave the angle must be less than 31.15°.

The classical wave with vorticity is Gerstner's extreme wave in deep water. Its profile is a cycloid with an angle of 90° at its cusp. There are numerous numerical calculations [14,7,15] showing that steady waves, including waves of finite depth, can be arbitrarily steep. The only known analytical bound on the angle is in the case of favorable vorticity, where an upper bound of 45° has been proved in [13].

In this paper, we prove for the first time that there is an upper bound on the angle even in some instances of adverse vorticity. (Adverse vorticity corresponds to positive vorticity in our sign convention.) We consider a connected set $\mathscr{C}$ of periodic symmetric steady water waves that contains at least one trivial wave. Denoting the fluid velocity by (U, V), the vorticity by γ, and the constant wave speed by c, we assume that $U < c$ and $|\gamma(U - c)| < g$ on the free surface S and that γ is non-increasing and concave. Then we prove in Section 2 that the slope of every streamline of every wave in $\mathscr{C}$ is less than $\frac{g-\gamma(U-c)}{g+\gamma(U-c)}$.

In Section 3 we consider the families of waves that were constructed using bifurcation theory in [4]. We prove that the slope of each of these waves is less than $\sqrt{1+\delta}$, provided that $\gamma > 0$ is less than a number $\bar{\gamma}$ that depends only on $\delta > 0$ and the given period L and flux m of the wave. In sections 4 and 5 we prove analogous results for the case of solitary waves, following work of Wheeler [17,18].

In his remarkable paper [1] Amick uses methods that are specific to the irrotational case and essentially do not generalize to other situations. For a brief discussion of Amick's methods, see [13]. New methods were developed in [13] to treat favorable vorticities of any magnitude. The purpose of the present paper is to treat some adverse vorticities by extending the methods of [13]. Some of our lemmas are basically repetitions of arguments in [13], but they are appropriately modified. These include some rather subtle applications of the Hopf maximum principle. On the other hand, a key new ingredient in the present paper is Lemma 2.3, which contains an analysis of the behavior of a wave at its point of maximum slope without assuming that the vorticity has a sign but assuming instead that gravity g dominates the vertical vortex force $|(U - c)\gamma|$ discussed in [10].

2. Bound on the slope of periodic waves

For the rest of this paper it is convenient to denote the relative velocity by

$$u = U - c\,, \quad v = V.$$

In this section and in Section 3, we consider symmetric $2L$-periodic water waves with vorticity. In a frame moving with the wave they are solutions of the system

$$uu_x + vu_y = -P_x \qquad \text{in } -d < y < \eta(x), \tag{2.1a}$$

$$uv_x + vv_y = -P_y - g \qquad \text{in } -d < y < \eta(x), \tag{2.1b}$$

$$u_x + v_y = 0 \qquad \text{in } -d < y < \eta(x), \tag{2.1c}$$

$$P = P_{\text{atm}} \qquad \text{on } y = \eta(x), \tag{2.1d}$$

$$v = \eta_x u \qquad \text{on } y = \eta(x), \tag{2.1e}$$

$$v = 0 \qquad \text{on } y = -d, \tag{2.1f}$$

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