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Continuation of homoclinic orbits in the suspension bridge equation: A computer-assisted proof

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Abstract

In this paper, we prove existence of symmetric homoclinic orbits for the suspension bridge equation $u'''' + \beta u'' + e^u - 1 = 0$ for all parameter values $\beta \in [0.5, 1.9]$. For each β, a parameterization of the stable manifold is computed and the symmetric homoclinic orbits are obtained by solving a projected boundary value problem using Chebyshev series. The proof is computer-assisted and combines the uniform contraction theorem and the radii polynomial approach, which provides an efficient means of determining a set, centered at a numerical approximation of a solution, on which a Newton-like operator is a contraction.

Keywords: Suspension bridge equation; Traveling waves; Contraction mapping; Rigorous numerics; Symmetric homoclinic orbits; Stable manifolds

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1. Introduction

One of the simplest models [15,12] for a suspension bridge is the partial differential equation (PDE)

$$\frac{\partial^2 U}{\partial T^2} = -\frac{\partial^4 U}{\partial X^4} - e^U + 1. \tag{1.1}$$

Here $U(T, X)$ describes the deflection of the roadway from the rest state $U = 0$ as a function of time T and the spatial variable X (in the direction of traffic). This paper is concerned with traveling wave solutions of (1.1), i.e., solutions $U(T, X) = u(X - cT)$ describing a disturbance with profile u propagating at velocity c along the surface of the bridge. In particular, we apply a computer-assisted proof method to show that there is a large range of velocities for which such a solitary wave exists.

Looking for traveling waves of (1.1) with wave speed c leads to the ordinary differential equation

$$u'''' + c^2 u'' + e^u - 1 = 0. \tag{1.2}$$

For large positive and negative values of the independent variable $t = X - cT$ we assume the solution to converge to the equilibrium $u = 0$. Due to the reversibility symmetry of the PDE in both time and space, we may restrict our attention to symmetric solutions. Hence, setting $\beta = c^2$, we are looking for symmetric homoclinic orbits satisfying

$$\begin{cases} u'''' + \beta u'' + e^u - 1 = 0 \\ u(-t) = u(t) \\ \lim_{t\to\infty} u(t) = 0. \end{cases} \tag{1.3}$$

Fourth order differential equations of the form $u'''' + \beta u'' + f(u) = 0$ for various nonlinearities f have been studied extensively. For the bistable nonlinearity $f(u) = u^3 - u$ the equation is a standard model in pattern formation, called the Swift-Hohenberg equation (see [18] and references therein), whereas the quadratic nonlinearity $f(u) = u^2 - u$ appears, for example, in the study of water waves [4]. For the piecewise linear case $f(u) = \max\{u, 0\}$ homoclinic solutions were obtained in [15,8]. For the problem with the exponential nonlinearity $f(u) = e^u - 1$ a family of periodic solutions was established in [17].

In [8] the question about existence of a symmetric homoclinic orbit of (1.3) is raised. This question was addressed by variational methods in [21], where the authors proved the result for *almost all* parameter values $\beta \in (0, 2)$. In [20] the existence of homoclinic orbits was demonstrated for all $\beta \in (0, c_*^2) \approx (0, 0.5516)$, again using variational methods as well as intricate estimates on the second variation. In a different direction, using a computer-assisted proof, it was proven in [3] that (1.3) has at least 36 homoclinic solutions for the single parameter value $\beta = 1.69$.

In the present paper we complement the above results by proving the following.

Theorem 1. *For all parameter values* $\beta \in [0.5, 1.9]$ *there exists a symmetric homoclinic orbit of* (1.3).

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