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Dichotomies for generalized ordinary differential equations and applications

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Abstract

In this work we establish the theory of dichotomies for generalized ordinary differential equations, introducing the concepts of dichotomies for these equations, investigating their properties and proposing new results. We establish conditions for the existence of exponential dichotomies and bounded solutions. Using the correspondences between generalized ordinary differential equations and other equations, we translate our results to measure differential equations and impulsive differential equations. The fact that we work in the framework of generalized ordinary differential equations allows us to manage functions with many discontinuities and of unbounded variation.

1. Introduction

The evolution of stability theory in differential equations depends, to a large extent, on results obtained for linear differential equations while the classical stability definitions given by

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Lyapunov are suitable for autonomous equations. However, non-autonomous systems require a more suitable notion of uniform stability: the exponential dichotomy.

The exponential dichotomy is a generalization of the concept of hyperbolicity of linear autonomous equations to linear non-autonomous differential equations, where the subspaces are replaced by invariant vector bundles and the stability properties of the solutions in these nontrivial invariant sets are uniform.

In the theory of ordinary differential equations (ODEs), the Hartman–Grobman linearization Theorem and the First Lyapunov Method are well-known. However, they are sufficient only for autonomous systems. K.J. Palmer, in [22], applies the exponential dichotomy to generalize the Hartman–Grobman Linearization Theorem for autonomous systems to non-autonomous systems and applies the exponential dichotomy technique to simplify the original proof of the Hartman–Grobman Linearization Theorem.

Detailed introductions about the theory of exponential dichotomy for ODEs can be found in Daleckii and Krein [8] and Coppel [7]. In recent years, K.J. Palmer had published a series of papers on exponential dichotomy for ODEs. For difference differential equations, the literature is more sparse, but Coffman and Schäfer [6] are the pioneers here. For impulsive differential equations (IDEs), the theory of exponential dichotomy can be found in Bainov et al. [4], for instance.

In the theory of non-autonomous dynamical systems, the importance of the theory of exponential dichotomy is due to the fact that it is widely used to solve nonlinear problems such as perturbation (see, e.g., Henry [13] and Sakamoto [24]). Further, since dichotomy is a kind of conditional stability, it can be used to investigate stability in modern chaos theory (Palmer [23]). In [19], the authors use the exponential dichotomy to characterize the structural stability of a vector field and give a proof of Smale's Conjecture showing that the theory of exponential dichotomy is also important in hyperbolic point set theory.

On the other hand, the interest in studying the theory of generalized ordinary differential equations (we will write generalized ODEs here) lies on the fact that these equations encompass various types of other differential equations such as ODEs, IDEs, measure differential equations (MDEs) and functional differential equations (see, e.g., [12], [16], [17], [18] and [21]). The reader may consult [1], [2], [3] and [11] to the study linking generalized ODEs with IDEs and retarded functional differential equations (RFDEs), and [10] and [26] for the study linking generalized ODEs and MDEs. In addition, RFDEs with variable time pulses can also be considered in the context of the generalized ODEs, see [1] and [2]. Thus, the context of the generalized ODEs is proving to be an excellent environment for dealing with problems of other classes of equations, especially when the functions involved have many discontinuities and/or are of unbounded variation. Furthermore, the generalized ODE environment is quite friendly, being simpler than any of the equations mentioned above.

The aim of the present paper is to describe the theory of dichotomies within generalized ODEs and apply the results to MDEs and IDEs.

This text is organized into four sections. In Section 2, we present some fundamental results on regulated functions, functions of bounded variation, the Kurzweil integral and the theory of generalized ODEs. We recall the basic concepts and main properties of generalized ODEs. Since the aim of this paper is to investigate dichotomies in the frame of linear generalized ODEs, we present, in Subsection 2.3, the fundamental theory for this class of linear equations.

Section 3 is devoted to the study of the theory of dichotomies for generalized ODEs. We present the concepts of ordinary and exponential dichotomies for linear generalized ODEs of the form