ARTICLE IN PRESS YJDEQ:9093

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Mayer control problem with probabilistic uncertainty on initial positions

Antonio Marigonda [a,*], Marc Quincampoix [b]

[a] *Department of Computer Science, University of Verona, Strada Le Grazie 15, I-37134 Verona, Italy*
[b] *Laboratoire de Mathématiques de Bretagne Atlantique, CNRS-UMR 6205, Université de Brest, 6, avenue Victor Le Gorgeu, CS 93837, 29238 Brest cedex 3, France*

Received 26 September 2017; revised 10 November 2017

Abstract

In this paper we introduce and study an optimal control problem in the Mayer's form in the space of probability measures on $\mathbb{R}^n$ endowed with the Wasserstein distance. Our aim is to study optimality conditions when the knowledge of the initial state and velocity is subject to some uncertainty, which are modeled by a probability measure on $\mathbb{R}^d$ and by a vector-valued measure on $\mathbb{R}^d$, respectively. We provide a characterization of the value function of such a problem as unique solution of an Hamilton–Jacobi–Bellman equation in the space of measures in a suitable viscosity sense. Some applications to a pursuit-evasion game with uncertainty in the state space is also discussed, proving the existence of a value for the game.

MSC: 49N70; 49L15

Keywords: Differential games; Optimal transport; Hamilton–Jacobi–Bellman equation

1. Introduction

We consider the following controlled differential equation

$$\dot{x}(t) = f(x(t), u(t)),\ u(t) \in U,\ t \in [0, T], \tag{1}$$

* Corresponding author.
E-mail addresses: antonio.marigonda@univr.it (A. Marigonda), marc.quincampoix@univ-brest.fr (M. Quincampoix).

https://doi.org/10.1016/j.jde.2017.11.014
0022-0396/

Please cite this article in press as: A. Marigonda, M. Quincampoix, Mayer control problem with probabilistic uncertainty on initial positions, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.11.014

where $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is a Lipschitz continuous function, the control set U is a compact subset of some finite dimensional vector space, and the control function $u(\cdot)$ is a Borel measurable function $u : [0, T] \mapsto U$.

The main features of the optimal control system we will investigate in the paper are the following:

- The initial position x_0 is not exactly known by the controller, but only a probabilistic description is available. More precisely, the initial state is described by a measure μ_0 with the following property: given any Borel set $A \subseteq \mathbb{R}^d$, the quantity $\mu_0(A)$ gives the probability that the initial position lies in the set A.
- Because of the uncertain initial position, to every point of the support of μ_0 there may correspond a possibly different choice of the control – hence a different possible velocity. Moreover we allow the "division of mass", i.e., even if the initial condition x_0 is known (namely $\mu_0 = \delta_{x_0}$), it can be split into different trajectories by several possible velocities in $f(x_0, U)$ but of course the total weight of these trajectories must remain equal to one.

So the natural state space of our control problem is the space $\mathscr{P}(\mathbb{R}^d)$ of Borel probability measures on $\mathbb{R}^d$. The conservation of mass along the corresponding trajectory $\boldsymbol{\mu} = \{\mu_t\}_{t\in[0,T]}$ (seen as a time-dependent probability measure), and the controlled dynamics, can be summarized in the following dynamical system

$$\begin{cases} \partial_t \mu_t + \operatorname{div}(v_t \mu_t) = 0, \\ \mu|_{t=0} = \mu_0, \\ v_t(x) \in F(x) := f(x, U), \quad \text{for } \mu_t\text{-almost every } x \in \mathbb{R}^d, \text{ a.e. } t \in [0, T], \end{cases} \tag{2}$$

where the first equation of the above system should be understood in the sense of distributions in $[0, T] \times \mathbb{R}^d$.

Observe that when $v_t(\cdot)$ is sufficiently regular (i.e., Lipschitz continuous), then the unique solution μ_t of (2) is the pushforward of the measure μ_0 by the flow at time t of the differential equation $\dot{x}(t) = v_t(x(t))$. We also note that the trajectories depend only on F and not on the specific parametrization $F(x) := f(x, U)$ and, consequently, we will mainly consider the differential inclusion $\dot{x}(t) \in F(x(t))$ whose trajectories are the same as those of (1).

We stress the fact that the measures μ_T that can be reached at time T from an initial measure μ by mean of an admissible trajectory in the sense of (2) are *not* simply the ones which are pushforward of the initial measure μ_0 by any Borel selection ϕ of the reachable set for the finite-dimensional underlying system. An example of this situation is provided by Example 2.10.

The controller aims to minimize the cost function depending on the value of trajectory at the terminal time T

$$\mathscr{J}(\boldsymbol{\mu}) := \mathscr{G}(\mu_T) \tag{3}$$

where, $\mathscr{G} : \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}$ is Lipschitz continuous with respect to the Wasserstein distance. A particular case of such a cost function is $\mathscr{G}_g(\mu) := \displaystyle\int_{\mathbb{R}^d} g(x)\, d\mu(x)$ where $g : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous and bounded. In this case, $\mathscr{G}_g$ turns out to be a Lipschitz map with respect to the Wasserstein distance on probability measures, and the final cost $\mathscr{G}_g(\mu_T)$ represents the expectation of the final cost g with respect to the probability measure μ_T. But such a cost is of moderate

Please cite this article in press as: A. Marigonda, M. Quincampoix, Mayer control problem with probabilistic uncertainty on initial positions, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.11.014