Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 264 (2018) 3402–3444

Journal of Differential Equations

www.elsevier.com/locate/jde

The Cauchy problem for the generalized Zakharov–Kuznetsov equation on modulation spaces

Tomoya Kato

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

Received 21 October 2016
Available online 19 December 2017

Abstract

We consider the Cauchy problem for the generalized Zakharov–Kuznetsov equation $\partial_t u + \partial_{x_1} \Delta u = \partial_{x_1}(u^{m+1})$ on three and higher dimensions. We mainly study the local well-posedness and the small data global well-posedness in the modulation space $M^0_{2,1}(\mathbb{R}^n)$ for $m \geq 4$ and $n \geq 3$. We also investigate the quartic case, i.e., $m = 3$.

Keywords: Generalized Zakharov–Kuznetsov equation; Well-posedness; Modulation spaces

1. Introduction

We study the Cauchy problem for the generalized Zakharov–Kuznetsov equation

$$\begin{cases} \partial_t u + \partial_{x_1} \Delta u = \partial_{x_1}(u^{m+1}), \\ u(0) = u_0, \end{cases} \tag{1}$$

where $u = u(x,t)$ is a real valued function, $t > 0$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $m \in \mathbb{N}$ and the Laplacian $\Delta = \partial^2_{x_1} + \cdots + \partial^2_{x_n}$. From the form of the first equation in (1), the generalized

E-mail address: t.katou@cr.math.sci.osaka-u.ac.jp.

https://doi.org/10.1016/j.jde.2017.11.020
0022-0396/

Zakharov–Kuznetsov equation can be seen as a multi-dimensional extension of the generalized Korteweg–de Vries (gKdV) equation on one dimension:

$$\partial_t u + \partial_x^3 u = \partial_x (u^{m+1}). \tag{2}$$

The Zakharov–Kuznetsov (ZK) equation, i.e., the first equation for $m = 1$ in (1), on three dimensions was invented by Zakharov and Kuznetsov [40] to describe propagation of ion-sound waves in magnetic fields. Also, the ZK equation on two dimensions was derived from the hydrodynamic set of equations for ion density and velocity by Laedke and Spatschek [21]. We remark that the ZK equation is justified rigorously only on two and three space dimensions (see [22]). However, in this paper, we consider the higher dimensional cases ($n \geq 4$) in addition to the justified dimensional case ($n = 3$) from a mathematical interest. In the following statements, we will sometimes write the gZK equation instead of the generalized Zakharov–Kuznetsov equation for short.

In this paper, we study the local well-posedness and the small data global well-posedness for the Cauchy problem (1) on three and higher dimensions in modulation spaces. Modulation spaces were originally introduced from the point of view of the time-frequency analysis by Feichtinger [8] in 1983. He first defined modulation spaces by using the short-time Fourier transform, which is one of techniques for the time-frequency analysis. Here, the short-time Fourier transform $V_g f(x, \omega)$ is denoted by

$$V_g f(x, \omega) = \int_{\mathbb{R}^n} e^{-it\cdot\omega} \overline{g(t - x)} f(t) dt$$

for a window function $g \in \mathcal{S} \setminus \{0\}$. We note that, if we choose a window function g satisfying that its support is compact and $g \equiv 1$ in a neighborhood of the origin, then the transform $V_g f$ enables us to understand relations between time and frequency of the function f. Now, let $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$. Then, the Feichtinger's original modulation space $M^s_{p,q}(\mathbb{R}^n)$ is defined by the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying that

$$\|f\|^{original}_{M^s_{p,q}} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \left| V_g f(x, \omega) \right|^p dx \right)^{q/p} \langle \omega \rangle^{sq} d\omega \right)^{1/q} \tag{3}$$

is finite. Here, $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and the norm (3) should be read with usual modification when p or q is infinity. Moreover, in [8], Feichtinger proved that the norm (3) can be equivalently expressed by the $\ell^q(L^p)$ norm. Let a Schwartz function sequence $\{\sigma_k\}_{k\in\mathbb{Z}^n}$ be

$$\operatorname{supp} \sigma \subset [-1, 1]^n, \quad \sigma_k = \sigma(\cdot - k) \quad \text{and} \quad \sum_{k\in\mathbb{Z}^n} \sigma_k(\xi) \equiv 1$$

for any $\xi \in \mathbb{R}^n$. Then, the norm (3) is equivalent to

$$\|f\|_{M^s_{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} \langle k \rangle^{sq} \|\Box_k f\|^q_{L^p} \right)^{1/q}$$