ARTICLE IN PRESS
YJDEQ:9102

Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations ••• (••••) •••–•••

Journal of Differential Equations

www.elsevier.com/locate/jde

Nonlinear Fourier transforms for the sine-Gordon equation in the quarter plane

Lin Huang, Jonatan Lenells *

Department of Mathematics, KTH Royal Institute of Technology, 100 44, Stockholm, Sweden

Received 18 October 2017

Abstract

Using the Unified Transform, also known as the Fokas method, the solution of the sine-Gordon equation in the quarter plane can be expressed in terms of the solution of a matrix Riemann–Hilbert problem whose definition involves four spectral functions a, b, A, B. The functions $a(k)$ and $b(k)$ are defined via a nonlinear Fourier transform of the initial data, whereas $A(k)$ and $B(k)$ are defined via a nonlinear Fourier transform of the boundary values. In this paper, we provide an extensive study of these nonlinear Fourier transforms and the associated eigenfunctions under weak regularity and decay assumptions on the initial and boundary values. The results can be used to determine the long-time asymptotics of the sine-Gordon quarter-plane solution via nonlinear steepest descent techniques.

MSC: 37K15; 41A60; 35P25

Keywords: Spectral function; Sine-Gordon equation; Inverse scattering; Initial-boundary value problem

Contents

* Corresponding author.
E-mail addresses: linhuang@kth.se (L. Huang), jlenells@kth.se (J. Lenells).

https://doi.org/10.1016/j.jde.2017.11.023

1. Introduction

The standard method for solving the initial-value problem for a *linear* partial differential equation (PDE) is to Fourier transform in space. Since the Fourier transform diagonalizes the differential operator, the time evolution in Fourier space is particularly simple and reduces to solving a linear ordinary differential equation for each frequency component.

In the context of *nonlinear* PDEs, the Fourier transform does not provide the same kind of simplification of the time evolution. Nonlinear PDEs can therefore not, in general, be analyzed in such a simple manner. However, there exists a class of nonlinear PDEs (called *integrable* PDEs) for which an analog of the above procedure exists. The first equation discovered to be integrable was the Korteweg–de Vries (KdV) equation [9], soon followed by the nonlinear Schrödinger [14] and sine-Gordon equations [1,11,15].

The class of integrable PDEs is set apart by the fact that it is possible to implement a 'nonlinear version' of the Fourier transform which reduces the time evolution to a set of linear ordinary differential equations. This nonlinear Fourier transform has to be tailor-made for each specific equation and takes, in general, a different form for each equation. For the initial value problem on the line, the transform is usually referred to as the Inverse Scattering Transform (IST) due to the fact that its implementation for the KdV equation involves a scattering problem for the Schrödinger equation in quantum mechanics.

The IST formalism provides a powerful framework for analyzing integrable nonlinear PDEs. For example, it is straightforward via IST techniques to construct large families of exact solutions, such as multi-soliton solutions. Soliton-generating techniques, being largely algebraic in nature, usually require a limited amount of analytical groundwork. On the other hand, the implementation of the IST relevant for the solution of the general initial value problem, or for the study of asymptotics, relies on detailed analytical estimates. In particular, estimates on the associated nonlinear Fourier transform and its inverse are required. To establish these estimates is a challenging and technical enterprise even for relatively simple cases. In the case of the KdV equation on the line, whose nonlinear Fourier transform is determined by the one-dimensional Schrödinger operator $-\partial_x^2 + q(x)$, Deift and Trubowitz put the IST analysis on a rigorous footing in the elegant but long paper [3].

An important development in recent years has been the extension of the IST formalism to initial-boundary value (IBV) problems, which is known as the Unified Transform or the Fokas method [4–6]. Although it is possible to treat more complicated domains, we here restrict attention to the case of IBV problems posed in the quarter-plane domain $\{x \geq 0, t \geq 0\}$, i.e., problems that involve a single boundary located at $x = 0$. Employing the machinery of [4–6], the solution of an integrable PDEs with a 2×2-matrix Lax pair in such a domain, can be expressed in terms of the solution of a matrix Riemann–Hilbert (RH) problem whose formulation involves four spectral functions $a(k)$, $b(k)$, $A(k)$, and $B(k)$. The functions $\{a(k), b(k)\}$ are defined via a nonlinear Fourier transform of the initial data on the half-line $\{x \geq 0, t = 0\}$, whereas $\{A(k), B(k)\}$ are defined via a nonlinear Fourier transform of the boundary values on $\{x = 0, t \geq 0\}$.

Please cite this article in press as: L. Huang, J. Lenells, Nonlinear Fourier transforms for the sine-Gordon equation in the quarter plane, J. Differential Equations (2017), https://doi.org/10.1016/j.jde.2017.11.023