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Global solvability and global hypoellipticity in Gevrey classes for vector fields on the torus

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Abstract

Let $L = \partial/\partial t + \sum_{j=1}^{N}(a_j + ib_j)(t)\partial/\partial x_j$ be a vector field defined on the torus $\mathbb{T}^{N+1} \simeq \mathbb{R}^{N+1}/2\pi\mathbb{Z}^{N+1}$, where a_j, b_j are real-valued functions and belonging to the Gevrey class $G^s(\mathbb{T}^1)$, $s > 1$, for $j = 1, \ldots, N$. We present a complete characterization for the s-global solvability and s-global hypoellipticity of L. Our results are linked to Diophantine properties of the coefficients and, also, connectedness of certain sublevel sets.

MSC: primary 35A01; secondary 35B10, 35B65

Keywords: Gevrey solvability; Gevrey hypoellipticity; Vector fields; Periodic solutions; Fourier series

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1. Introduction and statements of the results

For a positive integer m, let $\mathbb{T}^m \simeq \mathbb{R}^m/2\pi\mathbb{Z}^m$ be the m-dimensional torus. We will work on $\mathbb{T}^{N+1}$, where the coordinates are denoted by $(x,t) \in \mathbb{T}^N \times \mathbb{T}^1$, with $x = (x_1, \dots, x_N) \in \mathbb{T}^N$.

We say that a complex-valued function f is an s-Gevrey function on $\mathbb{T}^{N+1}$, $s \geq 1$, if f is C^∞ and there exist positive constants C and R such that, for all $\alpha \in \mathbb{Z}_+^{N+1}$ and all $(x,t) \in \mathbb{T}^{N+1}$, one has

$$|\partial^\alpha f(x,t)| \leq C R^{|\alpha|} \alpha!^s.$$

We denote by $G^s(\mathbb{T}^{N+1})$ the space of all s-Gevrey functions on $\mathbb{T}^{N+1}$.

In this paper we will make use of the following well-known characterizations of Gevrey functions. A complex-valued function $f(x,t)$ is an s-Gevrey function on $\mathbb{T}^{N+1}$ if f is C^∞ and there exist positive constants C, h and ϵ such that

$$|\partial_t^j \hat{f}(\xi,t)| \leq C h^j j!^s e^{-\epsilon|\xi|^{1/s}}, \quad \forall\, j \in \mathbb{N},\ \forall\, \xi \in \mathbb{Z}^N$$

(here $\hat{f}(\xi,t)$ denotes the ξ-th coefficient of the partial Fourier series of $f(x,t)$ in the x-variable). Also, $f(x,t)$ is an s-Gevrey function on $\mathbb{T}^{N+1}$ if f is C^∞ and there exist positive constants C and ϵ such that

$$|\hat{f}(\xi,\tau)| \leq C e^{-\epsilon(|\xi|+|\tau|)^{1/s}},\ \forall\, (\xi,\tau) \in \mathbb{Z}^{N+1}$$

(here $\hat{f}(\xi,\tau)$ denotes the (ξ,τ)-coefficient of the Fourier series of $f(x,t)$).

We are concerned with the existence and regularity of solutions in Gevrey classes, for $s > 1$, for operators $L : G^s(\mathbb{T}^{N+1}) \to G^s(\mathbb{T}^{N+1})$, which are given by

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} (a_j + i b_j)(t) \frac{\partial}{\partial x_j},\ (x,t) \in \mathbb{T}^N \times \mathbb{T}^1, \tag{1.1}$$

where a_j and b_j are real-valued functions belonging to $G^s(\mathbb{T}^1)$. Recall that the transpose operator of L, ${}^tL$, acts on the dual space $\mathcal{D}'_s(\mathbb{T}^{N+1})$ and

$$(\ker {}^tL)^\circ = \{\phi \in G^s(\mathbb{T}^{N+1});\ \langle \mu, \phi \rangle = 0,\ \text{for all } \mu \in \ker {}^tL \subset \mathcal{D}'_s(\mathbb{T}^{N+1})\}.$$

Note that if $f \in G^s(\mathbb{T}^{N+1})$ and $Lu = f$, for some $u \in G^s(\mathbb{T}^{N+1})$ then $f \in (\ker {}^tL)^\circ$; that is, $L(G^s(\mathbb{T}^{N+1})) \subset (\ker {}^tL)^\circ$.

We say that L is *s-globally solvable* if for every $f \in (\ker {}^tL)^\circ$ there exists $u \in G^s(\mathbb{T}^{N+1})$ solution to $Lu = f$.

We stress that we are interested in the solvability of the equation $Lu = f$ in the case where the function f and the coefficients of L are assumed in the same Gevrey class. Within this context this kind of problem was considered, for instance, in [1–3,8,11]; for the case where

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