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Bifurcation theory for finitely smooth planar autonomous differential systems

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Abstract

In this paper we establish bifurcation theory of limit cycles for planar C^k smooth autonomous differential systems, with $k \in \mathbb{N}$. The key point is to study the smoothness of bifurcation functions which are basic and important tool on the study of Hopf bifurcation at a fine focus or a center, and of Poincaré bifurcation in a period annulus. We especially study the smoothness of the first order Melnikov function in degenerate Hopf bifurcation at an elementary center. As we know, the smoothness problem was solved for analytic and C^∞ differential systems, but it was not tackled for finitely smooth differential systems. Here, we present their optimal regularity of these bifurcation functions and their asymptotic expressions in the finite smooth case.

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Keywords: C^k smooth system; Bifurcation function; Hopf bifurcation; Melnikov function; Smoothness; Limit cycle

1. Introduction

The second part of the Hilbert's 16th problem is on the maximum number $H(n)$ of limit cycles and their distribution of planar polynomial differential systems of a given degree n, see

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Hilbert [1]. As we know, it is still an open problem. Even for the quadratic systems, we only know $H(2) \geq 4$, see Chen and Wang [2], Shi [3]. Nowadays, a useful tool is to use the Melnikov function to study the number of limit cycles which are bifurcated from a center, or a period annulus, or a homoclinic loop of a Hamiltonian system under small perturbation, which is called the Melnikov function method. In applying this method, a key point is to know the regularity of the Melnikov functions.

Consider the near-Hamiltonian system

$$\begin{aligned} \dot{x} &= H_y(x,y) + \varepsilon f(x,y,\varepsilon), \\ \dot{y} &= -H_x(x,y) + \varepsilon g(x,y,\varepsilon), \qquad (x,y) \in \Omega \subset \mathbb{R}^2, \end{aligned} \tag{1}$$

where $\varepsilon \in \mathbb{R}$, Ω is an open domain and H, f, g are C^∞ or analytic functions. When H, f, g are polynomials, the problem on the maximum number of limit cycles of system (1) is called the weaken Hilbert's 16th problem Arnold [4]. It is still open also even for quadratic differential systems. See for instance Binyamini et al. [5], Chen et al. [6], Christopher and Li [7], Françoise [8], Hilbert [1], Roussarie [9], Aziz et al. [10], Colak et al. [11].

We now recall very briefly the Melnikov function method and its related results. For more information, see for instance Han and Zhu [12], Chen et al. [6], Han [13], Han et al. [14], Li et al. [15], Roussarie [9], Tian and Han [16], Varchenko [17], Wang and Yu [18], Xiong and Han [19], Zhang et al. [20,21], Zhao and Zhang [22], Xiong and Han [23]. Note that in many cases, system (1) contains a vector parameter $\delta \in D \subset R^m$ and when $\varepsilon = 0$ it has a family of periodic orbits, denoted by $L_h: H(x,y) = h$ with $h \in I$ an open interval. The main idea of the Melnikov function method is to establish a relationship between the number of limit cycles of system (1) with that of zeros of the first order Melnikov function as follows

$$M(h,\delta) = \oint_{L_h} g\,dx - f\,dy\Big|_{\varepsilon=0}.$$

It was proved that if for any $\delta \in D$, $M(h,\delta)$ has at most k zeros in h on I (taking into account multiplicity), then system (1) has at most k limit cycles bifurcating from the periodic orbits L_h for $|\varepsilon|$ small, see e.g. Han [24], Roussarie [9]. And if there exist $h_0 \in I$ and $\delta_0 \in D$ such that h_0 is a zero of $M(h,\delta_0)$ with odd multiplicity, then system (1) has at least one limit cycle near L_{h_0} for $|\varepsilon| + |\delta - \delta_0|$ small.

For the Hopf bifurcation, when α with L_α an elementary center is the left endpoint of I, Han Han [13] proved that if system (1) is C^∞, then $M(h,\delta)$ is a C^∞ function at $h = \alpha$ and it has the asymptotic expression

$$M(h,\delta) :\approx \sum_{i\geq 1} a_i(\delta)(h-\alpha)^i, \ 0 < h - \alpha \ll 1.$$

The number of limit cycles near the center L_α can be studied by using the coefficients a_i. For example, if there exist a $\delta_0 \in D$ and a $k \in \mathbb{N}$ such that $a_i(\delta_0) = 0$, $i = 1, \cdots, k$ and $a_{k+1}(\delta_0) \neq 0$, then system (1) has at most k limit cycles near the center L_α for $|\varepsilon| + |\delta - \delta_0| \ll 1$. Moreover, if

$$\operatorname{rank} \frac{\partial(a_1,\cdots,a_k)}{\partial\delta}\Bigg|_{\delta=\delta_0} = k,$$

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