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Non-localization of eigenfunctions for Sturm–Liouville operators and applications

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Abstract

In this article, we investigate a non-localization property of the eigenfunctions of Sturm–Liouville operators $A_a = -\partial_{xx} + a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions, where $a(\cdot)$ runs over the bounded nonnegative potential functions on the interval $(0, L)$ with $L > 0$. More precisely, we address the extremal spectral problem of minimizing the L^2-norm of a function $e(\cdot)$ on a measurable subset ω of $(0, L)$, where $e(\cdot)$ runs over all eigenfunctions of A_a, at the same time with respect to all subsets ω having a prescribed measure and all L^∞ potential functions $a(\cdot)$ having a prescribed essentially upper bound. We provide some existence and qualitative properties of the minimizers, as well as precise lower and upper estimates on the optimal value. Several consequences in control and stabilization theory are then highlighted.

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1. Introduction

1.1. Localization/non-localization of Sturm–Liouville eigenfunctions

In a recent survey article concerning the Laplace operator ([10]), D. Grebenkov and B.T. Nguyen introduce, recall and gather many possible definitions of the notion of *localization of eigenfunctions*. In particular, in section 7.7 of their article, they consider the Dirichlet–Laplace operator Δ on a given bounded open set Ω of $\mathbb{R}^n$, a Hilbert basis of eigenfunctions $(e_j)_{j\in\mathbb{N}^*}$ in $L^2(\Omega)$ and use as a measure of localization of the eigenfunctions on a measurable subset $\omega \subset \Omega$ the following criterion

$$C_p(\omega) = \inf_{j\in\mathbb{N}^*} \frac{\|e_j\|^p_{L^p(\omega)}}{\|e_j\|^p_{L^p(\Omega)}},$$

where $p \geqslant 1$. For instance, evaluating this quantity for different choices of subdomains ω if Ω is a ball or an ellipse allows to illustrate the so-called *whispering galleries* or *bouncing ball* phenomena. At the opposite, when Ω denotes the d-dimensional box $(0, \ell_1) \times \cdots \times (0, \ell_d)$ (with $\ell_1, \dots, \ell_d > 0$), it is recalled that $C_p(\omega) > 0$ for any $p \geqslant 1$ and any measurable subset $\omega \subset \Omega$ whenever the ratios $(\ell_i/\ell_j)^2$ are not rational numbers for every $i \neq j$.

Many other notions of localization have been introduced in the literature. Regarding the Dirichlet/Neumann/Robin Laplacian eigenfunctions on a bounded open domain Ω of $\mathbb{R}^n$ and using a semi-classical analysis point of view, the notions of *quantum limit* or *entropy* have been widely investigated (see e.g. [1,3,4,9,13,20]) and provide an account for possible strong concentrations of eigenfunctions. Notice that the properties of $C_p(\omega)$ are intimately related to the behavior of high-frequency eigenfunctions and especially to the set of quantum limits of the sequence of eigenfunctions considered. Identifying such limits is a great challenge in quantum physics ([4,9,40]) and constitute a key ingredient to highlight non-localization/localization properties of the sequence of eigenfunctions considered.

Given a nonzero integer p, the *non-localization property* of a sequence $(e_j)_{j\in\mathbb{N}^*}$ of eigenfunctions means that the real number $C_p(\omega)$ is positive for every measurable subset $\omega \subset \Omega$. Concerning the one-dimensional Dirichlet–Laplace operator on $\Omega = (0, \pi)$, it has been highlighted in the case where $p = 2$ (for instance in [12,24,36]) that

$$\inf_{|\omega|=r\pi} C_2(\omega) = \inf_{|\omega|=r\pi} \inf_{j\in\mathbb{N}^*} \frac{2}{\pi} \int_\omega \sin(jx)^2\, dx > 0,$$

for every $r \in (0, 1)$.

Motivated by these considerations, the present work is devoted to studying similar issues in the case $p = 2$, for a general family of one-dimensional Sturm–Liouville operators of the kind $A_a = -\partial_{xx} + a(\cdot)\,\mathrm{Id}$ with Dirichlet boundary conditions, where $a(\cdot)$ is a nonnegative essentially bounded potential defined on the interval $(0, L)$. More precisely, we aim at providing lower